

On Twistors and Conformal Field Theories from Six Dimensions

Christian Sämann^a and Martin Wolf^b *

^a *Maxwell Institute for Mathematical Sciences
Department of Mathematics, Heriot–Watt University
Edinburgh EH14 4AS, United Kingdom*

^b *Department of Mathematics, University of Surrey
Guildford GU2 7XH, United Kingdom*

Abstract

We discuss chiral zero-rest-mass field equations on six-dimensional space-time from a twistorial point of view. Specifically, we present a detailed cohomological analysis, develop both Penrose and Penrose–Ward transforms, and analyse the corresponding contour integral formulæ. We also give twistor space action principles. We then dimensionally reduce the twistor space of six-dimensional space-time to obtain twistor formulations of various theories in lower dimensions. Besides well-known twistor spaces, we also find a novel twistor space amongst these reductions, which turns out to be suitable for a twistorial description of self-dual strings. For these reduced twistor spaces, we explain the Penrose and Penrose–Ward transforms as well as contour integral formulæ.

11th November 2011

To the memory of Francis A. Dolan

* *E-mail addresses:* c.saemann@hw.ac.uk, m.wolf@surrey.ac.uk

Contents

1	Introduction	1
2	Spinors and free fields in six dimensions	4
2.1	Spinors in six dimensions	4
2.2	Zero-rest-mass fields in six dimensions	5
3	Twistor space of six-dimensional space-time	8
3.1	Twistor space from space-time	8
3.2	Space-time from twistor space	11
4	Penrose transform in six dimensions	13
4.1	Cohomological considerations	15
4.2	Proof	22
4.3	Integral formulæ and thickenings	26
5	Penrose–Ward transform in six dimensions	29
6	Reduction to lower dimensions	38
6.1	Field equations in lower dimensions	39
6.2	Ambitwistors and Maxwell fields	41
6.3	Hyperplane twistors and self-dual strings	45
6.4	Minitwistors and monopoles	52
7	Conclusions and outlook	53
	Appendix	56
A	Spinor notation and reality conditions in six and four dimensions	56
B	Some remarks about n -gerbes	59

1. Introduction

Since their discovery by Penrose [1], twistors have provided deep insights into various gauge and gravity theories, particularly into integrable ones. The cornerstone of twistor geometry is to replace space-time as a background for physical processes by an auxiliary space called twistor space. Differentially constrained data (such as solutions to field equations on space-time) are then encoded in differentially unconstrained complex analytic data (such as elements of cohomology groups) on twistor space. This allows for an elegant and complete description of solutions to certain classes of problems. The prime examples in this respect are all the solutions to the zero-rest-mass field equations on four-dimensional space-time [1–4], instantons in Yang–Mills theory [5], and self-dual Riemannian four-dimensional manifolds [6]. Twistor geometry moreover underlies the well-known Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction of Yang–Mills instanton solutions [7]. Even solutions to non-linear second-order differential equations such as the full Yang–Mills and Einstein

equations and their supersymmetric extensions can be captured in terms of holomorphic data using twistor methods [8–13]. In these latter cases, however, the power to explicitly construct solutions is limited.

Perhaps the simplest and oldest example of a twistor description of space-time is obtained by replacing Minkowski space $\mathbb{R}^{1,3}$ by the space of projective light cones in this space, $\mathbb{R}^{1,3} \times S^2$. This space can be shown to be diffeomorphic to the open subset $\mathbb{P}^3_\circ := \mathbb{P}^3 \setminus \mathbb{P}^1$ of complex projective three-space \mathbb{P}^3 .¹ In this paper, we will always work with complexified space-times. The twistor description of space-time \mathbb{C}^4 is given by the space of totally null two-planes, which is again \mathbb{P}^3_\circ . Moreover, holomorphic vector bundles over \mathbb{P}^3_\circ , that are subject to a mild triviality condition, are in one-to-one correspondence with Yang–Mills instantons on \mathbb{C}^4 . There is a large variety of further examples of twistor spaces, on each of which there are such correspondences between cohomological data and solutions to field equations (via the so-called Penrose and Penrose–Ward transforms). In this paper, we shall encounter Penrose’s twistor space [1], the ambitwistor space [8, 9], and Hitchin’s minitwistor space [14]. For detailed reviews of various aspects of twistor spaces, see for example [15–17]. See also [18, 19] for recent reviews using conventions close to ours.

All the above-mentioned twistor spaces are suitable for capturing moduli spaces² of the four-dimensional Yang–Mills equations, their supersymmetric extensions, the BPS subsectors thereof and their dimensional reductions.³ In the light of the recent success of M2-brane models [23], it is natural to wonder about twistor spaces underlying the description of solution spaces of more general gauge theories. Staying within M-theory, there are essentially three theories one might be interested in. The three-dimensional M2-brane models, M5-brane models which should be given by some $\mathcal{N} = (0, 2)$ superconformal field theories in six dimension, and the self-dual string equation in four dimensions which describes a BPS subsector of the dimensionally reduced M5-brane model.

In this paper, we shall focus on the latter two. The self-dual string equation given by Howe, Lambert & West [24] is a BPS equation that describes M2-branes ending on M5-branes. It can be seen as the M-theory analogue of the Bogomolny monopole equation describing D1-branes ending on D3-branes. Moreover, this equation has recently been shown to be integrable when re-phrased on loop space by establishing an ADHM-like

¹One may also conformally compactify space-time to obtain \mathbb{P}^3 as twistor space. The line \mathbb{P}^1 that is deleted from \mathbb{P}^3 to obtain \mathbb{P}^3_\circ corresponds on space-time to the point infinity which is used for this conformal compactification.

²These moduli spaces are obtained from the solution spaces by quotienting with respect to the group of gauge transformations.

³Examples of twistor spaces for higher-dimensional space-times including Penrose and Penrose–Ward transforms can be found e.g. in [20–22].

construction [25]. This is a strong hint that a twistor description of the solutions and the moduli space of this equation should be possible. Since the self-dual string equation arises from a reduction of the six-dimensional theory of self-dual three-forms, one should first consider a twistor space describing such three-forms and then further reduce it. Fortunately, a candidate twistor space that is suitable for a twistorial description of chiral theories has already appeared [15, 16, 21, 26–30]. We shall denote this twistor space by P^6 . Since the non-Abelian extensions of both the self-dual string equation and the self-dual three-forms are still essentially unknown⁴, we shall restrict ourselves to the Abelian cases. The hope is certainly that the twistor descriptions presented below might shed light on the issue of non-Abelian extensions.

Our first aim is to establish both Penrose and Penrose–Ward transforms for the construction of chiral zero-rest-mass spinor fields on six-dimensional space-time using the twistor space P^6 . In particular, we shall give a detailed proof of a Penrose transform to establish an isomorphism between certain cohomology groups on P^6 and chiral zero-rest-mass fields on space-time. Our discussion follows the corresponding one in the four-dimensional case as given, e.g. in [32, 17].⁵ Moreover, we also show how to generalise the Penrose–Ward transform to P^6 and how to obtain spinor fields via this transform. We shall also introduce twistor space action principles for chiral fields which might be the twistor analog of the space-time actions of Pasti, Sorokin & Tonin [33].

Our second aim is to demonstrate how the dimensional reductions of the six-dimensional spinor fields to four and three space-time dimensions is reflected in certain reductions of P^6 . In particular, we find that the twistor space P^6 contains naturally the ambitwistor space, which provides a twistor description of the Maxwell and Yang–Mills equations, a twistor space we shall refer to as the hyperplane twistor space and which turns out to be suitable for a twistor description of the self-dual string equation, and the minitwistor space underlying a twistor description of monopoles. To our knowledge, the hyperplane twistor space has not been discussed in the literature before. Therefore, we shall be explicit in constructing both Penrose and Penrose–Ward transforms over this twistor space.

This paper is structured as follows. We begin our considerations with a brief review of spinors and free fields in six dimensions. We then present the construction of the twistor space for six-dimensional space-time from various perspectives in Section 3. In Section 4, we lay down the cohomological foundation on which all of our later analysis is based. This section also contains a detailed proof of the Penrose transform and explicit integral

⁴There is a non-Abelian extension of the self-dual string equation on loop space [25]. Also there have been some recent proposals for non-Abelian M5-brane models, see e.g. [31].

⁵An alternative proof can be found in [21].

formulae yielding zero-rest-mass fields. The Penrose–Ward transform is presented in Section 5, where we also comment on the aforementioned action principle. We then continue with discussing various dimensional reductions in Section 6. In particular, we show how the six-dimensional picture reduces to the ambitwistor space describing Maxwell fields in four dimensions, the twistor description of self-dual strings and the twistor description of monopoles. We summarise our conclusions in Section 7, where we also present an outlook. Two appendices collect some technical background material.

Remark. Whilst finalising the draft, we became aware of the results of Mason, Reid-Edwards & Taghavi-Chabert [34], which partially overlap with the results presented in this work.

Dedication. We would like to dedicate this work to our friend and colleague Francis A. Dolan, who passed away very unexpectedly in September this year.

2. Spinors and free fields in six dimensions

2.1. Spinors in six dimensions

In the following, we shall be working with the complexification of flat six-dimensional space-time $M^6 := \mathbb{C}^6$. Notice that reality conditions leading to real slices of M^6 with Minkowski or split signature can be imposed if desired. These are briefly discussed in Appendix A.

The spin bundle on M^6 is of rank eight and decomposes into the direct sum $S \oplus \tilde{S}$ of the two rank-4 subbundles of anti-chiral spinors, S , and chiral spinors, \tilde{S} . There is a natural isomorphism identifying S and \tilde{S} with the duals⁶ \tilde{S}^\vee and S^\vee (see e.g. Penrose & Rindler [16] for details; this identification basically works via an automorphism of the Clifford algebra corresponding to charge conjugation). Therefore, we may exclusively work with, say, S and S^\vee . We shall label the corresponding spinors by upper and lower capital Latin letters from the beginning of the alphabet, e.g. ψ^A for a section of S and ψ_A for a section of S^\vee , with $A, B, \dots = 1, \dots, 4$.

We may identify the tangent bundle T_{M^6} with the anti-symmetric tensor product of the chiral spinor bundle with itself via

$$\begin{aligned} T_{M^6} &\cong S \wedge S, \\ \partial_M &:= \frac{\partial}{\partial x^M} \xleftrightarrow{\tilde{\sigma}_*} \partial_{AB} := \frac{\partial}{\partial x^{AB}}. \end{aligned} \tag{2.1}$$

⁶Given a linear space V , we denote its dual by V^\vee .

Here, we coordinatised M^6 by x^M , for $M, N, \dots = 1, \dots, 6$ and used the identification $\tilde{\sigma} : x = (x^M) \mapsto \tilde{\sigma}(x) = (x^{AB})$ with $x^{AB} = \tilde{\sigma}_M^{AB} x^M \Leftrightarrow x^M = \frac{1}{4} \sigma_{AB}^M x^{AB}$, where $\tilde{\sigma}_M^{AB}$, σ_{AB}^M are the six-dimensional sigma-matrices, cf. Appendix A. The induced linear mapping $\tilde{\sigma}_*$ is explicitly given as $\partial_{AB} = \frac{1}{4} \sigma_{AB}^M \partial_M$ and the (flat) metric η_{MN} on M^6 can be identified with the Levi-Civita symbol $\frac{1}{2} \varepsilon_{ABCD}$ in spinor notation. Hence, $\sigma_{AB}^M = \frac{1}{2} \varepsilon_{ABCD} \tilde{\sigma}^{MCD}$ and we can raise and lower indices according to

$$\partial_{AB} = \frac{1}{2} \varepsilon_{ABCD} \partial^{CD} \quad \Longleftrightarrow \quad \partial^{AB} = \frac{1}{2} \varepsilon^{ABCD} \partial_{CD} . \quad (2.2)$$

For any two six-vectors $p = (p^M)$ and $q = (q^M)$, we shall write

$$p \cdot q := p_M q^M = \frac{1}{4} p_{AB} q^{AB} = \frac{1}{8} \varepsilon_{ABCD} p^{AB} q^{CD} , \quad (2.3)$$

and we have $p^2 := p \cdot p = \sqrt{\det p^{AB}}$.

2.2. Zero-rest-mass fields in six dimensions

Next we wish to discuss zero-rest-mass fields in six-dimensional spinor-helicity formalism, borrowing some of the ideas of [35]. Let us start by considering a momentum six-vector $p = (p_M)$. If we impose the null-condition $p^2 = 0$, then we have $\det p_{AB} = 0 = \det p^{AB}$. These equations are solved most generally by

$$p_{AB} = k_{Aa} k_{Bb} \varepsilon^{ab} \quad \text{and} \quad p^{AB} = \tilde{k}^{A\dot{a}} \tilde{k}^{B\dot{b}} \varepsilon_{\dot{a}\dot{b}} \quad (2.4)$$

with $a, b, \dots, \dot{a}, \dot{b}, \dots = 1, 2$ and $\varepsilon^{ab} = -\varepsilon^{ba}$ and $\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}}$. We shall refer to such a momentum as null-momentum. Moreover, transformations of the form $k_{Aa} \mapsto M_a^b k_{Ab}$ and $\tilde{k}^{A\dot{a}} \mapsto \tilde{M}^{\dot{a}}_{\dot{b}} \tilde{k}^{A\dot{b}}$ with $\det M = 1 = \det \tilde{M}$ will leave p invariant, which shows that the indices a, \dot{a}, \dots are little group indices. The little group of (complex) null-vectors in six dimensions is therefore $\text{SL}(2, \mathbb{C}) \times \widetilde{\text{SL}(2, \mathbb{C})}$. Notice that $k_{Aa} \tilde{k}^{A\dot{b}} = 0$ since $p_{AB} = \frac{1}{2} \varepsilon_{ABCD} p^{CD}$, which, in turn, shows that k_{Aa} and $\tilde{k}^{A\dot{a}}$ are not independent. Notice also that k_{Aa} has $4 \times 2 = 8$ components, but three of them can be fixed by little group transformations. Thus, k_{Aa} has indeed exactly the five independent components needed to describe the (five-dimensional) null-cone in six dimensions.

Fields form irreducible representations of the Lorentz group which are induced from representations of the little group. In six dimensions, the spin label of fields therefore has to be generalised to a pair of integers, labelling the irreducible representations of the little group $\text{SL}(2, \mathbb{C}) \times \widetilde{\text{SL}(2, \mathbb{C})}$. As an example of zero-rest-mass fields, let us consider the fields in the $\mathcal{N} = (0, 2)$ tensor multiplet. This multiplet is a chiral multiplet and hence the fields transform trivially under the $\widetilde{\text{SL}(2, \mathbb{C})}$ subgroup. Amongst these fields, there is a

self-dual three-form $H = dB$, which transforms as the $(\mathbf{3}, \mathbf{1})$ of the little group. In spinor notation, H has components⁷ $H_{AB} = \partial_{C(A} B_{B)}^C$, where B_B^C is trace-less and denotes the components of a two-form potential B in spinor notation. In addition, we have four Weyl spinors ψ_A^I in the $(\mathbf{2}, \mathbf{1})$ and five scalars ϕ^{IJ} in the trivial representation $(\mathbf{1}, \mathbf{1})$ of the little group. Notice that the a priori six components of $\phi^{IJ} = -\phi^{JI}$ are reduced to five by the condition $\phi^{IJ}\Omega_{IJ} = 0$, where $I, J, \dots = 1, \dots, 4$ and Ω_{IJ} is an invariant form of the underlying R -symmetry; see e.g. [36] for more details. In the following, we shall work with complex fields. However, one may impose reality conditions on all fields as briefly discussed in Appendix A. The zero-rest-mass field equations (i.e. the free equations of motion) for the fields in the tensor multiplet read as

$$H^{AB} = 0 \quad \text{with} \quad \partial^{AC} H_{CB} = 0, \quad \partial^{AB} \psi_B = 0 \quad \text{and} \quad \square \phi = 0, \quad (2.5)$$

where we suppressed the R -symmetry indices. Notice that the second equation is the Bianchi identity (which, of course, is equivalent to the field equation for self-dual three-forms). The corresponding plane waves are given by the expressions ($i := \sqrt{-1}$)

$$H_{ABab} = k_{A(a} k_{Bb)} e^{ix \cdot p}, \quad \psi_{Aa} = k_{Aa} e^{ix \cdot p} \quad \text{and} \quad \phi = e^{ix \cdot p}. \quad (2.6)$$

This follows from straightforward differentiation. Here, the representations of the little group formed by the fields become explicit. Furthermore, since $H_{AB} = \partial_{C(A} B_{B)}^C$, we can express the plane waves of H_{AB} in terms of the plane waves of the potential two-form B_B^A . To this end, we note that in spinor notation, gauge transformations of B_B^A are mediated by gauge parameters $\Lambda_{AB} = \Lambda_{[AB]}$ via $B_B^A \mapsto B_B^A + \partial^{AC} \Lambda_{CB} - \partial_{BC} \Lambda^{CA}$. We shall choose Lorenz gauge, which in spinor notation reads as $\partial_{C[A} B_{B]}^C = 0 = \partial^{C[A} B_{C}^{B]}$. The residual gauge transformations are given by gauge parameters that obey $\partial \cdot \Lambda = 0$. Let us now choose reference spinors μ_{Aa} and define the null-momentum $q_{AB} := \mu_{Aa} \mu_{Bb} \varepsilon^{ab}$ so that $p \cdot q \neq 0$. Then the plane waves of the potential two-form B_B^A in Lorenz gauge are given by

$$B_B^A{}_{ab} = \kappa_{(a}^A k_{Bb)} e^{ix \cdot p} \quad \text{with} \quad \kappa_a^A := -2i \frac{q^{AB} k_{Ba}}{p \cdot q}. \quad (2.7)$$

Clearly, B_B^A is trace-less and one can check that $\partial_{C(A} B_{B)}^C$ yields the components for H_{AB} given in (2.6). Since $\partial^{CA} B_C^B = 0$, we also have $H^{AB} = \partial^{C(A} B_{C}^{B)} = 0$, which implies that B_B^A does indeed yield a self-dual field strength. Furthermore, the choice of μ_{Aa} is

⁷ A general three-form $H = dB$ in six dimensions is described by a pair of symmetric bi-spinors $H = (H_{AB}, H^{AB}) = (\partial_{C(A} B_{B)}^C, \partial^{C(A} B_{C}^{B)})$ and transforms as the $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$ of the little group. We use parentheses and square brackets to denote normalised symmetrisation and normalised anti-symmetrisation, respectively. By imposing either self-duality or anti-self-duality, one of the bi-spinors is put to zero. In our conventions, self-duality implies $H^{AB} = 0$ while anti-self-duality amounts to $H_{AB} = 0$.

irrelevant since changes in μ_{Aa} merely correspond to (residual) gauge transformations of B_B^A , a fact that is already familiar from four dimensions [37].⁸ One may analyse other spin fields in a very similar way and we shall present a few more comments in Remark 2.1 below.

Let us now introduce some notions and notation that we shall make use of in this paper. We shall mostly be interested in *chiral zero-rest-mass fields*, i.e. fields forming representations $(\mathbf{2s} + \mathbf{1}, \mathbf{1})$, $s \in \frac{1}{2}\mathbb{N}_0$, of the little group $\mathrm{SL}(2, \mathbb{C}) \times \widetilde{\mathrm{SL}(2, \mathbb{C})}$. These fields will carry $2s$ symmetrised spinor indices. Specifically, using the conventions

$$\begin{aligned} [k] &:= \otimes^k \det S^\vee, \quad [-k] := [k]^\vee \quad \text{and} \quad [0] := [k] \otimes [-k] \quad \text{for } k \in \mathbb{N}, \\ \mathcal{S}[\pm k] &:= \mathcal{S} \otimes_{\mathcal{O}_{M^6}} [\pm k] \quad \text{for some Abelian sheaf } \mathcal{S} \text{ on } M^6, \end{aligned} \quad (2.8)$$

we shall denote the sheaf of chiral zero-rest-mass fields on M^6 by \mathcal{Z}_s ,

$$\mathcal{Z}_s := \begin{cases} \ker \{ \partial^{AB} : (\odot^{2s} S^\vee)[1] \rightarrow (\odot^{2s-1} S^\vee \otimes_{\mathcal{O}_{M^6}} S)_0[2] \} & \text{for } s \geq \frac{1}{2}, \\ \ker \{ \square := \frac{1}{4} \partial^{AB} \partial_{AB} : [1] \rightarrow [2] \} & \text{for } s = 0. \end{cases} \quad (2.9)$$

Here, the subscript zero refers to the totally trace-less part. The factors $[\pm k]$ are referred to as *conformal weights*, as they render the zero-rest-mass field equations conformally invariant.

Remark 2.1. Recall that there is a potential formulation of zero-rest-mass fields in four dimensions, cf. e.g. Ward & Wells [17]. This formulation generalises to six dimensions, as we shall demonstrate now. Consider an $s \in \frac{1}{2}\mathbb{N}^*$. From the potential fields

$$B_A^{A_1 \cdots A_{2s-1}} = B_A^{(A_1 \cdots A_{2s-1})} \in H^0(M^6, (\odot^{2s-1} S \otimes_{\mathcal{O}_{M^6}} S^\vee)_0[1]), \quad (2.10)$$

we derive a field strength $H_{A_1 \cdots A_{2s}} \in H^0(U, (\odot^{2s} S^\vee)[1])$ according to

$$H_{A_1 \cdots A_{2s}} := \partial_{(A_1 B_1} \cdots \partial_{A_{2s-1} B_{2s-1}} B_{A_{2s}}^{B_1 \cdots B_{2s-1}}. \quad (2.11)$$

The equations

$$H^{A_1 \cdots A_{2s}} := \partial^{A(A_1} B_{A_1}^{A_2 \cdots A_{2s})} = 0 \quad (2.12)$$

then imply that

$$\partial^{AA_1} H_{A_1 \cdots A_{2s}} = 0. \quad (2.13)$$

⁸Since the μ_{Aa} appear only in the combination $q_{AB} = \mu_{Aa} \mu_{Bb} \varepsilon^{ab}$, we may focus on the induced changes in q . The space of the q is five-dimensional, so the most general change is of the form $q_{AB} \mapsto q_{AB} + \alpha q_{AB} + \beta_{AB}$ with $\beta^2 = 0$ and $\beta \cdot p = 0$. Therefore, $B_B^A{}_{ab} \mapsto B_B^A{}_{ab} + \Lambda_{(a}^A k_{Bb)} e^{ix \cdot p}$, with $\Lambda_a^A := -2i \frac{\beta^{AB} k_{Ba}}{(1+\alpha)p \cdot q}$. This is just a (residual) gauge transformation as a consequence of $\beta \cdot p = 0$.

Furthermore, the pair of spinors $(H_{A_1 \dots A_{2s}}, H^{A_1 \dots A_{2s}})$ is invariant under gauge transformations of the form

$$B_B^{AA_1 \dots A_{2s-2}} \mapsto B_B^{AA_1 \dots A_{2s-2}} + \left[\partial_{CB} \Lambda^{C(AA_1 \dots A_{2s-2})} - \partial^{C(A} \Lambda_{CB}^{A_1 \dots A_{2s-2})} \right]_0, \quad (2.14)$$

where the subscript zero refers again to the totally trace-less part and $\Lambda_{AB}^{A_1 \dots A_{2s-2}} = \Lambda_{[AB]}^{(A_1 \dots A_{2s-2})}$ is totally trace-less itself. Note that the traces of $[\partial_{CB} \Lambda^{C(AA_1 \dots A_{2s-2})} - \partial^{C(A} \Lambda_{CB}^{A_1 \dots A_{2s-2})}]$ always drop out of the above definition of $(H_{A_1 \dots A_{2s}}, H^{A_1 \dots A_{2s}})$. Altogether, the spinor field $H_{A_1 \dots A_{2s}}$ can therefore be regarded as a section of the sheaf \mathcal{Z}_s . We shall make use of this formulation in Section 5 when dealing with the Penrose–Ward transform.

3. Twistor space of six-dimensional space-time

In this section, we shall review a particular twistor space associated with M^6 that is a very natural generalisation of known twistor spaces and suitable for the description of chiral theories in six dimensions. This twistor space has appeared earlier e.g. in [15, 16, 21, 26–30]. Here we shall present a detailed discussion of its constructions from an alternative point of view.

Remark 3.1. *We shall always be working with locally free sheaves and therefore we shall not make a notational distinction between vector bundles and their corresponding sheaves of sections. We shall switch between the two notions freely depending on context.*

3.1. Twistor space from space-time

Let us consider the projectivisation $\mathbb{P}(S^\vee)$ of the dual chiral spin bundle S^\vee . Since S^\vee is of rank four, $\mathbb{P}(S^\vee) \rightarrow M^6$ is a \mathbb{P}^3 -bundle over M^6 . Hence, the projectivisation $\mathbb{P}(S^\vee)$ is a nine-dimensional complex manifold $F^9 \cong \mathbb{C}^6 \times \mathbb{P}^3$, the⁹ *correspondence space*. We take $(x, \lambda) = (x^{AB}, \lambda_A)$ as coordinates on F^9 , where λ_A are homogeneous coordinates on \mathbb{P}^3 .

Consider now the following vector fields on F^9 :

$$V^A := \lambda_B \frac{\partial}{\partial x_{AB}}. \quad (3.1)$$

Note that $\lambda_A V^A = 0$ because of the anti-symmetry of the spinor indices in the partial derivative. These vector fields define an integrable rank-3 distribution on F^9 , which we call *twistor distribution*. Therefore, we have a foliation of F^9 by three-dimensional complex

⁹The reason for this name will become transparent momentarily.

manifolds. The resulting quotient will be *twistor space*, a six-dimensional manifold denoted by P^6 . We have thus established the following double fibration:

$$\begin{array}{ccc} & F^9 & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ P^6 & & M^6 \end{array} \quad (3.2)$$

Let $(z, \lambda) = (z^A, \lambda_A)$ be homogeneous coordinates on \mathbb{P}^7 and assume that $\lambda_A \neq 0$. This effectively means that we are working on the open subset

$$\mathbb{P}^7_{\circ} := \mathbb{P}^7 \setminus \mathbb{P}^3 \quad (3.3)$$

of \mathbb{P}^7 , where the removed \mathbb{P}^3 is given by $z^A \neq 0$ and $\lambda_A = 0$. In the double fibration (3.2), the projection π_2 is the trivial projection and $\pi_1 : (x^{AB}, \lambda_A) \mapsto (z^A, \lambda_A) = (x^{AB}\lambda_B, \lambda_A)$. Thus, P^6 forms a quadric hypersurface inside \mathbb{P}^7_{\circ} , which is given by the equation

$$z^A \lambda_A = 0. \quad (3.4)$$

We shall refer to the relation

$$z^A = x^{AB} \lambda_B \quad (3.5)$$

as *incidence relation*, because it is a direct generalisation of Penrose's incidence relation in four dimensions. Notice that (3.2) makes also clear why F^9 is called correspondence space: It links space-time with twistor space via a double fibration.

Geometric twistor correspondence. The double fibration (3.2) shows that points in either of the base spaces M^6 and P^6 correspond to subspaces of the other base space: For any point $x \in M^6$, the corresponding manifold $\hat{x} := \pi_1(\pi_2^{-1}(x)) \hookrightarrow P^6$ is a three-dimensional complex manifold bi-holomorphic to \mathbb{P}^3 as follows from (3.5). Conversely, for any fixed $p = (z, \lambda) \in P^6$, the most general solution to the incidence relation (3.5) is given by

$$x^{AB} = x_0^{AB} + \varepsilon^{ABCD} \mu_C \lambda_D, \quad (3.6)$$

where x_0^{AB} is a particular solution and μ_A is arbitrary. This defines a totally null-plane $\pi_2(\pi_1^{-1}(p))$ in M^6 . This plane is three-dimensional because of the freedom in the choice of μ_A given by the shifts $\mu_A \mapsto \mu_A + \varrho \lambda_A$ for $\varrho \in \mathbb{C}$ which do not alter the solution (3.6).

Altogether, points in space-time correspond to complex projective three-spaces in twistor space while points in twistor space correspond to totally null three-planes in space-time. Thus, twistor space parametrises all totally null three-planes of space-time.

Twistor space as a normal bundle. The above considerations imply that P^6 can be viewed as a holomorphic vector bundle over \mathbb{P}^3 , where the global holomorphic sections are given by the incidence relation (3.5). In fact, (3.5) shows that P^6 is a rank-3 subbundle of the bundle $\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \rightarrow \mathbb{P}^3$, whose total space is \mathbb{P}^7_\circ . Here and in the following, $\mathcal{O}_{\mathbb{P}^3}(1)$ denotes the dual tautological bundle¹⁰ over \mathbb{P}^3 .

To identify the subbundle P^6 , let us denote by $N_{Y|X}$ the normal bundle of some complex submanifold Y of a complex manifold X , $i : Y \hookrightarrow X$. This bundle is defined by the following short exact sequence:

$$0 \longrightarrow T_Y \longrightarrow i^*T_X \longrightarrow N_{Y|X} \longrightarrow 0. \quad (3.7)$$

Let us now specialise to $Y = \mathbb{P}^3$ and $X = \mathbb{P}^7$ with coordinates (z^A, λ_A) on \mathbb{P}^7 as before. If $\mathbb{P}^3 \hookrightarrow \mathbb{P}^7$ is given by $z^A = 0$ and $\lambda_A \neq 0$, then $T_{\mathbb{P}^3} = \langle \frac{\partial}{\partial \lambda_A} \rangle$ and $T_{\mathbb{P}^7} = \langle \frac{\partial}{\partial z^A}, \frac{\partial}{\partial \lambda_A} \rangle$. The normal bundle of $N_{\mathbb{P}^3|\mathbb{P}^7}$ of \mathbb{P}^3 inside \mathbb{P}^7 is given by

$$0 \longrightarrow T_{\mathbb{P}^3} \longrightarrow i^*T_{\mathbb{P}^7} \longrightarrow N_{\mathbb{P}^3|\mathbb{P}^7} \longrightarrow 0, \quad (3.8a)$$

which implies that

$$N_{\mathbb{P}^3|\mathbb{P}^7} \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4, \quad (3.8b)$$

since the coefficient functions of the basis vector fields $\frac{\partial}{\partial z^A}$ and $\frac{\partial}{\partial \lambda_A}$ are linear in the coordinates. Hence, the z^A can be regarded as fibre coordinates of $N_{\mathbb{P}^3|\mathbb{P}^7}$, while the λ_A are base coordinates.

Using these results, we find that our twistor space P^6 fits into the short exact sequence¹¹

$$0 \longrightarrow P^6 \longrightarrow N_{\mathbb{P}^3|\mathbb{P}^7} \xrightarrow{\kappa} \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow 0, \quad (3.9a)$$

where

$$\kappa : (z^A, \lambda_A) \mapsto z^A \lambda_A. \quad (3.9b)$$

Note that the sequence (3.9a) can be regarded as an alternative definition of twistor space. Again, we see that P^6 is a rank-3 subbundle of $\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \rightarrow \mathbb{P}^3$ as stated earlier. It also shows that P^6 is the normal bundle of \mathbb{P}^3 inside the quadric hypersurface $\mathbb{Q}^6 \hookrightarrow \mathbb{P}^7$ given by the zero locus $z^A \lambda_A = 0$. Moreover, notice that the open subset $\mathbb{Q}^6 \cap \mathbb{P}^7_\circ$ can be identified with P^6 .

¹⁰or hyperplane bundle

¹¹We use the notation $\mathcal{O}_{\mathbb{P}^3}(k) := \otimes^k \mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{O}_{\mathbb{P}^3}(-k) := \mathcal{O}_{\mathbb{P}^3}^\vee(k)$, $k > 0$, as well as $\mathcal{O}_{\mathbb{P}^3}(0) = \mathcal{O}_{\mathbb{P}^3}$.

3.2. Space-time from twistor space

Next we wish to address the problem of how to obtain space-time M^6 , and in particular the factorisation (2.1) of the tangent bundle, from twistor space using (3.9a). To this end, consider the long exact sequence of cohomology groups induced by the short exact sequence (3.9a),

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}^3, P^6) \longrightarrow H^0(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}) \xrightarrow{\kappa} H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \\ \longrightarrow H^1(\mathbb{P}^3, P^6) \longrightarrow H^1(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}) \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow \cdots, \end{aligned} \quad (3.10)$$

where we have slightly abused notation by again using the letter κ . To compute these cohomology groups, we recall a special case of the Borel–Weil–Bott theorem:

Lemma 3.1. (*Bott’s rule [38]*) *Let V be an n -dimensional complex vector space. Consider its projectivisation $\mathbb{P}(V)$ together with the hyperplane bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$. Furthermore, set $\mathcal{O}_{\mathbb{P}(V)}(k) := \otimes^k \mathcal{O}_{\mathbb{P}(V)}(1)$, $\mathcal{O}_{\mathbb{P}(V)}(-k) := \mathcal{O}_{\mathbb{P}(V)}^\vee(k)$ and $\mathcal{O}_{\mathbb{P}(V)}(0) := \mathcal{O}_{\mathbb{P}(V)}$ for $k \in \mathbb{N}$. Then*

$$H^q(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(k)) \cong \begin{cases} \odot^k V^\vee & \text{for } q = 0 \text{ \& } k \geq 0, \\ \odot^{-k-n} V \otimes \det V & \text{for } q = n - 1 \text{ \& } k \leq -n, \\ 0 & \text{otherwise,} \end{cases} \quad (3.11)$$

where $\det V \equiv \Lambda^n V$.

From Bott’s rule for $V = \mathbb{C}^4$, we find that $H^1(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}) = 0 = H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ and furthermore

$$H^1(\mathbb{P}^3, P^6) = 0, \quad (3.12)$$

since κ is surjective. Therefore, the long exact sequence of cohomology groups (3.10) reduces to

$$0 \longrightarrow H^0(\mathbb{P}^3, P^6) \longrightarrow H^0(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}) \xrightarrow{\kappa} H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \longrightarrow 0. \quad (3.13)$$

By applying Bott’s rule again, we deduce from the latter sequence that

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^3, P^6) = 6. \quad (3.14)$$

Because of (3.12) and (3.14), we may now apply Kodaira’s theorem of relative deformation theory¹² to conclude that there is a six-dimensional family of deformations of \mathbb{P}^3 inside the quadric hypersurface $\mathbb{Q}^6 \hookrightarrow \mathbb{P}^7$. We shall denote this family by M^6 .

¹²Recall that Kodaira’s theorem states that if Y is a compact complex submanifold of a not necessarily compact complex manifold X with $H^1(Y, N_{Y|X}) = 0$, then there exists a $\dim_{\mathbb{C}} H^0(Y, N_{Y|X})$ -dimensional family of deformations of Y inside X . For more details, see e.g. [39].

Next we define the correspondence space F^9 according to

$$F^9 := \{(p, x) \in P^6 \times M^6 \mid p \in \hat{x}\} . \quad (3.15)$$

Notice that F^9 is fibred over both P^6 and M^6 . The typical fibres of $\pi_2 : F^9 \rightarrow M^6$ are complex projective three-spaces \mathbb{P}^3 . Hence, we have again established a double fibration of the form (3.2), where the fibres of $F^9 \rightarrow P^6$ are three-dimensional complex submanifolds of M^6 .

On F^9 , we may consider the relative tangent bundle, denoted by T_{π_1} , along the fibration $\pi_1 : F^9 \rightarrow P^6$. It is of rank three and defined by

$$0 \longrightarrow T_{\pi_1} \longrightarrow T_{F^9} \longrightarrow \pi_1^* T_{P^6} \longrightarrow 0 . \quad (3.16)$$

By construction, the vector fields V^A given in (3.1) annihilate $z^A = x^{AB} \lambda_B$ and therefore, T_{π_1} can be identified with the twistor distribution. Hence, sections μ_A of T_{π_1} are defined up to shifts by terms proportional to λ_A (recall that $\lambda_A V^A = 0$). Then we define a bundle N on F^9 by

$$\begin{aligned} 0 &\longrightarrow T_{\pi_1} \longrightarrow \pi_2^* T_{M^6} \longrightarrow N \longrightarrow 0 , \\ \mu_A &\mapsto \varepsilon^{ABCD} \mu_C \lambda_D , \\ \xi^{AB} &\mapsto \xi^{AB} \lambda_B . \end{aligned} \quad (3.17)$$

Clearly, the rank of N is three and the restriction of N to the fibre $\pi_2^{-1}(x)$ of $F^9 \rightarrow M^6$ for $x \in M^6$ is isomorphic to the pull-back $\pi_1^* N_{\hat{x}|P^6}$ of the normal bundle $N_{\hat{x}|P^6}$ of $\hat{x} \hookrightarrow P^6$. Thus, N can be identified with $\pi_1^* N_{\hat{x}|P^6}$.¹³

These considerations allow us to reconstruct the tangent bundle T_{M^6} from twistor space. In fact, we may apply the direct image functor (with regard to π_2) to the short exact sequence (3.17). Since both direct images¹⁴ $\pi_{2*} T_{\pi_1}$ and $\pi_{2*}^1 T_{\pi_1}$ vanish, we obtain

$$T_{M^6} \cong \pi_{2*} \pi_1^* N_{\hat{x}|P^6} \iff (T_{M^6})_x \cong H^0(\hat{x}, N_{\hat{x}|P^6}) . \quad (3.18)$$

Elements of $H^0(\hat{x}, N_{\hat{x}|P^6})$ are given in terms of elements of $H^0(\mathbb{P}^3, P^6)$ by allowing the latter to depend on x . One can check that this dependence is holomorphic in an open neighbourhood of x .

What remains to be understood is how the explicit factorisation (2.1) of the tangent bundle emerges from the above construction and in particular from $H^0(\mathbb{P}^3, P^6)$. To show

¹³Note that $N_{\hat{x}|P^6}$ is bi-holomorphic to $P^6 \rightarrow \mathbb{P}^3$.

¹⁴Remember that the q -th direct image sheaf $\pi_*^q \mathcal{S}$ of some Abelian sheaf \mathcal{S} on X for some map $\pi : X \rightarrow Y$ is defined by the pre-sheaf $Y \supset U$ open $\mapsto H^q(\pi^{-1}(U), \mathcal{S})$. We abbreviate $\pi_* \mathcal{S} := \pi_*^0 \mathcal{S}$. See also Section 4 for details of the computation of direct image sheaves, in particular Proposition 4.2.

this, we consider the Euler sequence for \mathbb{P}^3 ,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \longrightarrow T_{\mathbb{P}^3} \longrightarrow 0. \quad (3.19)$$

Upon dualising this sequence and twisting by $\mathcal{O}_{\mathbb{P}^3}(2)$, we find

$$0 \longrightarrow T_{\mathbb{P}^3}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow 0. \quad (3.20)$$

By comparing with (3.9a), we may conclude that¹⁵

$$P^6 \cong \Omega^1(2) \quad \text{with} \quad \Omega^p(k) := \Omega_{\mathbb{P}^3}^p \otimes \mathcal{O}_{\mathbb{P}^3}(k). \quad (3.21)$$

Thus, elements of $H^0(\mathbb{P}^3, P^6)$ can also be viewed as elements of $H^0(\mathbb{P}^3, \Omega^1(2))$. The latter are of the form $\omega = \omega^{AB} \lambda_A d\lambda_B$ with $\omega^{AB} = -\omega^{BA}$. Since

$$S_x \cong H^0(\hat{x}, \mathcal{O}_{\hat{x}}(1)) \quad (3.22)$$

via $s^A \mapsto s^A \lambda_A$ for $s^A \in S_x$, we indeed find the factorisation $(T_{M^6})_x \cong S_x \wedge S_x$. This concludes our construction of space-time from twistor space.

Remark 3.2. Notice that an identification of the form (2.1) amounts to choosing a (holomorphic) conformal structure. This can be seen as follows: Let X be a six-dimensional complex spin manifold. The first definition of a conformal structure on X (and perhaps the standard one) assumes an equivalence class $[g]$, the conformal class, of holomorphic metrics g on X . Two given metrics g and g' are called equivalent if $g' = \gamma^2 g$ for some nowhere vanishing holomorphic function γ . Thus, a conformal structure is a line subbundle L in $T_X^\vee \odot T_X^\vee$.

An alternative definition of a conformal structure assumes a factorisation of the form $T_X \cong S \wedge S$, where S is the rank-4 chiral spin bundle. This isomorphism in turn gives (canonically) the line subbundle $\det S^\vee \equiv \Lambda^4 S^\vee$ in $T_X^\vee \odot T_X^\vee$ since upon using splitting principle arguments (see e.g. [40]), one finds the identification $K_X := \det T_X^\vee \cong \otimes^3 \det S^\vee$ for the canonical bundle K_X . Hence, $\det S^\vee$ can be identified with the line bundle L from above, and the metric g is then of the form $\gamma^2 \varepsilon_{ABCD}$.

4. Penrose transform in six dimensions

Having defined twistor space, we would like to understand differentially constrained data on space-time in terms of differentially unconstrained data on twistor space. Specifically,

¹⁵Here and in the following, we shall denote the one-forms on some manifold X by either T_X^\vee or by Ω_X^1 .

we are interested in the chiral fields introduced in Section 2.2 and prove the following theorem:¹⁶

Theorem 4.1. *Consider the double fibration (3.2). Let $U \subset M^6$ be open and convex and set $U' := \pi_2^{-1}(U) \subset F^9$ and $\hat{U} := \pi_1(\pi_2^{-1}(U)) \subset P^6$, respectively. For $h \in \frac{1}{2}\mathbb{N}_0$, there is a canonical isomorphism*

$$\mathcal{P} : H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)) \rightarrow H^0(U, \mathcal{Z}_s) , \quad (4.1)$$

where $s = h$. This transformation is called the Penrose transform.

As familiar from four dimensions, one can write down integral formulæ which make this Penrose transform explicit. We shall do this in Section 4.3, after proving the above theorem. Comparing with the four-dimensional case, one would expect a Penrose transform for all $h \in \frac{1}{2}\mathbb{Z}$. However, we shall see below that the integral formulæ for $h < 0$ suggest that it is necessary to consider *infinitesimal extensions* of the cohomology groups on P^6 to the ambient space $\mathbb{P}^7_{\circ} = \mathbb{P}^7 \setminus \mathbb{P}^3$ in order to get zero-rest-mass fields. This is familiar from the construction of solutions to the four-dimensional Maxwell and Yang–Mills equations using the so-called ambitwistor space, and we shall come back to this point later. Because of this issue, we shall not develop the Penrose transform for negative h in this way, but rather deal with this case by establishing certain Penrose–Ward transforms in Section 5. To simplify the discussion, we shall restrict ourselves to complex holomorphic fields which correspond to real-analytic solutions upon imposing reality conditions as briefly discussed in Appendix A.

There are various steps involved in proving the above theorem and the structure of the proof will be similar to the one given in the four-dimensional setting. Therefore, the reader might find it useful to consult additionally e.g. the article by Eastwood, Penrose & Wells [32] or the text book by Ward & Wells [17] and references therein, where the four-dimensional case is presented in detail. We also refer to Buchdahl [41] and Pool [42] for very good accounts on various cohomological constructions which we shall make use of below. Notice that Murray [21] has given a proof for Penrose transforms on twistor spaces of certain even-dimensional Riemannian manifolds. Here, we shall present a very detailed proof¹⁷ from a different point of view, which is closer to [32] or [17] and which can be transferred relatively straightforwardly to our discussion presented in Section 6.3.

The subsequent section provides the cohomological foundations needed for proving the above Penrose transform and the Penrose–Ward transforms in Section 5. In particular, we

¹⁶We shall use the notation $\mathcal{O}_{P^6}(k) := \text{pr}^* \mathcal{O}_{\mathbb{P}^3}(k)$ where $\text{pr} : P^6 \rightarrow \mathbb{P}^3$ is the bundle projection and likewise for the open sets.

¹⁷Among other things, we shall e.g. carefully track conformal weights.

shall introduce the so-called relative de Rham complex on the correspondence space of the double fibration (3.2) and compute its cohomology by applying the direct image functor with respect to the projection π_2 . We shall also recall a result of Buchdahl's which allows us to pull-back cohomological data from the twistor space to the correspondence space. All these ingredients eventually enable us to establish isomorphisms between cohomological data on the twistor space and (chiral) spinor fields on space-time.

4.1. Cohomological considerations

Relative de Rham complex. The starting point of our considerations is the double fibration (3.2). As a first tool in proving the Penrose transform, we introduce the relative differential forms $\Omega_{\pi_1}^p$, i.e. the differential p -forms along the fibres of the fibration $\pi_1 : F^9 \rightarrow P^6$. We have already introduced the corresponding relative tangent bundle in (3.16). Simply dualising this sequence, we obtain the definition of the sheaf of relative one-forms from

$$0 \longrightarrow \pi_1^* \Omega_{P^6}^1 \longrightarrow \Omega_{F^9}^1 \longrightarrow \Omega_{\pi_1}^1 \longrightarrow 0. \quad (4.2)$$

Recall from our previous discussion that in our parametrisation, sections μ_A of the relative tangent bundle T_{π_1} are defined up to shifts by terms proportional to λ_A . This, in turn, induces the condition $\omega^A \lambda_A = 0$ on sections ω^A of $\Omega_{\pi_1}^1$. We shall come back to this point when discussing the direct images of $\Omega_{\pi_1}^1$.

In general, we introduce the relative p -forms $\Omega_{\pi_1}^p$ on F^9 with respect to the fibration $\pi_1 : F^9 \rightarrow P^6$ according to

$$0 \longrightarrow \pi_1^* \Omega_{P^6}^1 \wedge \Omega_{F^9}^{p-1} \longrightarrow \Omega_{F^9}^p \longrightarrow \Omega_{\pi_1}^p \longrightarrow 0. \quad (4.3)$$

Thus, relative p -forms have components only along the fibres of $\pi_1 : F^9 \rightarrow P^6$ (i.e. any contraction with a vector field perpendicular to $T_{\pi_1} \hookrightarrow T_{F^9}$ vanishes). The coefficient functions in local coordinates, however, depend on both the base and the fibre coordinates. Note that the maximum value of p here is three. If we let $\text{pr}_{\pi_1} : \Omega_{F^9}^p \rightarrow \Omega_{\pi_1}^p$ be the quotient mapping, we can define the relative exterior derivative d_{π_1} by setting

$$d_{\pi_1} := \text{pr}_{\pi_1} \circ d : \Omega_{\pi_1}^p \rightarrow \Omega_{\pi_1}^{p+1}, \quad (4.4)$$

where d is the usual exterior derivative on F^9 . In local coordinates (x^{AB}, λ_A) on F^9 , the relative exterior derivative can be presented in terms of the vector fields (3.1).

Next, observe that the relative differential d_{π_1} induces the *relative de Rham complex*. This complex is given in terms of an injective resolution of the topological inverse¹⁸ $\pi_1^{-1} \mathcal{O}_{P^6}$

¹⁸Remember that the topological inverse $\pi^{-1} \mathcal{S}$ of some Abelian sheaf \mathcal{S} on Y for $\pi : X \rightarrow Y$ is defined by the pre-sheaf $X \supset U \text{ open} \mapsto H^0(\pi(U), \mathcal{S})$.

of \mathcal{O}_{P^6} on the correspondence space F^9 :

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_{P^6} \longrightarrow \mathcal{O}_{F^9} \xrightarrow{d\pi_1} \Omega_{\pi_1}^1 \xrightarrow{d\pi_1} \Omega_{\pi_1}^2 \xrightarrow{d\pi_1} \Omega_{\pi_1}^3 \longrightarrow 0. \quad (4.5)$$

We shall not explain here why this sequence is exact but instead refer the interested reader to Ward & Wells [17] or Buchdahl [41], where more general discussions are given.

A natural question is now if the sheaves $\Omega_{\pi_1}^p$ have an interpretation in terms of certain pull-back sheaves from space-time and twistor space. Notice that the vectors fields (3.1) are given by $V^A = \frac{1}{2}\varepsilon^{ABCD}\lambda_B\partial_{CD}$, where ∂_{AB} are the vector fields spanning T_{M^6} . In terms of the V^A , the map $d\pi_1 : \mathcal{O}_{F^9} \rightarrow \Omega_{\pi_1}^1$ reads explicitly as

$$V^A : f \mapsto \omega^A = V^A f = \frac{1}{2}\varepsilon^{ABCD}\lambda_B\partial_{CD}f, \quad f \in \mathcal{O}_{F^9}. \quad (4.6)$$

This shows that $\omega^A = V^A f$ is a section of $\pi_2^*(\det S^\vee \otimes_{\mathcal{O}_{M^6}} S) \otimes_{\mathcal{O}_{F^9}} \pi_1^*\mathcal{O}_{P^6}(1)$. Clearly, it is not the most general section of this sheaf, since we have $\lambda_A\omega^A = \lambda_A V^A f = 0$; see also our comments given below (4.2). For a general section s^A of $\pi_2^*(\det S^\vee \otimes_{\mathcal{O}_{M^6}} S) \otimes_{\mathcal{O}_{F^9}} \pi_1^*\mathcal{O}_{P^6}(1)$, the map $\lambda_A : s^A \mapsto s^A\lambda_A$ gives a section of $\pi_2^*\det S^\vee \otimes_{\mathcal{O}_{F^9}} \pi_1^*\mathcal{O}_{P^6}(2)$ and its kernel gives $\Omega_{\pi_1}^1$. Altogether, we conclude that $\Omega_{\pi_1}^1$ fits into the following short exact sequence:

$$\begin{aligned} 0 \longrightarrow \Omega_{\pi_1}^1 \longrightarrow \pi_2^*(\det S^\vee \otimes_{\mathcal{O}_{M^6}} S) \otimes_{\mathcal{O}_{F^9}} \pi_1^*\mathcal{O}_{P^6}(1) \longrightarrow \\ \longrightarrow \pi_2^*\det S^\vee \otimes_{\mathcal{O}_{F^9}} \pi_1^*\mathcal{O}_{P^6}(2) \longrightarrow 0. \end{aligned} \quad (4.7)$$

Using the notation (2.8), we then obtain the following proposition:

Proposition 4.1. *The sheaves appearing in the relative de Rham sequence (4.5) can be canonically identified as follows. With $\Omega_{\pi_1}^p(k) := \Omega_{\pi_1}^p \otimes_{\mathcal{O}_{F^9}} \pi_1^*\mathcal{O}_{P^6}(k)$, we have*

$$0 \longrightarrow \Omega_{\pi_1}^p \longrightarrow \pi_2^*(\Lambda^p S)[p] \otimes_{\mathcal{O}_{F^9}} \pi_1^*\mathcal{O}_{P^6}(p) \longrightarrow \pi_2^*[1] \otimes_{\mathcal{O}_{F^9}} \Omega_{\pi_1}^{p-1}(2) \longrightarrow 0. \quad (4.8)$$

Proof: Using the fact that short exact sequences of the form $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{L} \rightarrow 0$, where \mathcal{L} is the sheaf of sections of some line bundle, always induce $0 \rightarrow \Lambda^p \mathcal{E} \rightarrow \Lambda^p \mathcal{F} \rightarrow \Lambda^{p-1} \mathcal{E} \otimes \mathcal{L} \rightarrow 0$, the sequence (4.7) immediately leads to (4.8). \square

Finally, we point out that the relative de Rham sequence (4.5) has a natural extension via twisting by a holomorphic vector bundle. Specifically, let $E \rightarrow P^6$ be a holomorphic vector bundle over P^6 and consider the pull-back bundle π_1^*E over the correspondence space F^9 . We may tensor (4.5) by $\pi_1^{-1}\mathcal{O}_{P^6}(E)$, which is the sheaf of sections of π_1^*E that are constant along $\pi_1 : F^9 \rightarrow P^6$. Because $\mathcal{O}_{F^9}(\pi_1^*E) \cong \pi_1^*\mathcal{O}_{P^6}(E)$ and $\mathcal{O}_{F^9} \otimes_{\pi_1^{-1}\mathcal{O}_{P^6}} \pi_1^{-1}\mathcal{O}_{P^6}(E)$ are canonically isomorphic, we find

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_{P^6}(E) \longrightarrow \Omega_{\pi_1}^0(E) \xrightarrow{d\pi_1} \dots \xrightarrow{d\pi_1} \Omega_{\pi_1}^3(E) \longrightarrow 0, \quad (4.9a)$$

where we have defined

$$\Omega_{\pi_1}^0(E) := \mathcal{O}_{F^9}(\pi_1^*E) \quad \text{and} \quad \Omega_{\pi_1}^p(E) := \Omega_{\pi_1}^p \otimes_{\mathcal{O}_{F^9}} \mathcal{O}_{F^9}(\pi_1^*E) . \quad (4.9b)$$

Direct image sheaves. The next important ingredient for our subsequent discussion is the direct images of $\Omega_{\pi_1}^p(E)$ with respect to the fibration $\pi_2 : F^9 \rightarrow M^6$ for the special case $E = \mathcal{O}_{P^6}(k)$, $k \in \mathbb{Z}$. To compute those, we shall make use of the following lemma:

Lemma 4.1. *Let V be a four-dimensional complex vector space together with its projectivisation $\mathbb{P}(V)$. Using the shorthand notations $\Omega^p(k) := \Omega_{\mathbb{P}(V)}^p \otimes \mathcal{O}_{\mathbb{P}(V)}(k)$ and $\Omega^0(k) := \mathcal{O}_{\mathbb{P}(V)}(k)$, we have the following list of sheaf cohomology groups:*

$$H^q(\mathbb{P}(V), \Omega^0(k)) \cong \begin{cases} \odot^k V^\vee & \text{for } q = 0 \text{ \& } k \geq 0 , \\ \odot^{-k-4} V \otimes \det V & \text{for } q = 3 \text{ \& } k \leq -4 , \\ 0 & \text{otherwise ,} \end{cases} \quad (4.10a)$$

$$H^q(\mathbb{P}(V), \Omega^1(k)) \cong \begin{cases} \left[\frac{\odot^{k-1} V \otimes V}{\odot^k V} \right]^\vee & \text{for } q = 0 \text{ \& } k \geq 2 , \\ \mathbb{C} & \text{for } q = 1 \text{ \& } k = 0 , \\ V^\vee \otimes \det V & \text{for } q = 3 \text{ \& } k = -3 , \\ \left[\frac{\odot^{-k-3} V^\vee \otimes V}{\odot^{-k-4} V^\vee} \right]^\vee \otimes \det V & \text{for } q = 3 \text{ \& } k < -3 , \\ 0 & \text{otherwise ,} \end{cases} \quad (4.10b)$$

$$H^q(\mathbb{P}(V), \Omega^2(k)) \cong \begin{cases} V \otimes \det V^\vee & \text{for } q = 0 \text{ \& } k = 3 , \\ \frac{\odot^{k-3} V^\vee \otimes V}{\odot^{k-4} V^\vee} \otimes \det V^\vee & \text{for } q = 0 \text{ \& } k > 3 , \\ \mathbb{C} & \text{for } q = 2 \text{ \& } k = 0 , \\ \frac{\odot^{-k-1} V \otimes V}{\odot^{-k} V} & \text{for } q = 3 \text{ \& } k \leq -2 , \\ 0 & \text{otherwise ,} \end{cases} \quad (4.10c)$$

$$H^q(\mathbb{P}(V), \Omega^3(k)) \cong \begin{cases} \odot^{k-4} V^\vee \otimes \det V^\vee & \text{for } q = 0 \text{ \& } k \geq 4, \\ \odot^{-k} V & \text{for } q = 3 \text{ \& } k \leq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4.10d)$$

Notice that here, we are essentially computing the Dolbeault cohomology groups $H_{\bar{\partial}}^{p,q}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$ of the complex projective three-space \mathbb{P}^3 with values in $\mathcal{O}_{\mathbb{P}^3}(k)$ via the Dolbeault isomorphism.

Proof: We already know the cohomology groups (4.10a) from Bott's rule given in Lemma 3.1. Moreover, after computing (4.10b), all remaining cases follow directly from (4.10a) and (4.10b) via Serre duality.¹⁹ In fact, we find the cohomology groups (4.10c) and (4.10d) from

$$\begin{aligned} H^q(\mathbb{P}(V), \Omega^2(k)) &\cong [H^{3-q}(\mathbb{P}(V), \Omega^1(-k))]^\vee, \\ H^q(\mathbb{P}(V), \Omega^3(k)) &\cong [H^{3-q}(\mathbb{P}(V), \Omega^0(-k))]^\vee. \end{aligned} \quad (4.11)$$

To compute (4.10b), let us consider the Euler sequence (3.19). We can dualise this sequence and twist by $\mathcal{O}_{\mathbb{P}(V)}(k)$ to obtain

$$0 \longrightarrow \Omega^1(k) \longrightarrow \Omega^0(k-1) \otimes V^\vee \longrightarrow \Omega^0(k) \longrightarrow 0. \quad (4.12)$$

From this sequence and Bott's rule, we derive the long exact sequences of cohomology groups

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}(V), \Omega^1(k)) \longrightarrow H^0(\mathbb{P}(V), \Omega^0(k-1) \otimes V^\vee) &\xrightarrow{\kappa} \\ &\xrightarrow{\kappa} H^0(\mathbb{P}(V), \Omega^0(k)) \longrightarrow H^1(\mathbb{P}(V), \Omega^1(k)) \longrightarrow 0, \end{aligned} \quad (4.13a)$$

and

$$\begin{aligned} 0 \longrightarrow H^3(\mathbb{P}(V), \Omega^1(k)) \longrightarrow H^3(\mathbb{P}(V), \Omega^0(k-1) \otimes V^\vee) &\longrightarrow \\ &\longrightarrow H^3(\mathbb{P}(V), \Omega^0(k)) \longrightarrow 0, \end{aligned} \quad (4.13b)$$

where we used $H^2(\mathbb{P}(V), \Omega^1(k)) = 0$.

Let us start with $H^q(\mathbb{P}(V), \Omega^1(k))$ for $q = 0, 1$. For $k < 0$, the sequence (4.13a) together with Bott's rule yield that $H^0(\mathbb{P}(V), \Omega^1(k)) = 0 = H^1(\mathbb{P}(V), \Omega^1(k))$ while for $k = 0$ we find $H^0(\mathbb{P}(V), \Omega^1(0)) = 0$ and $H^1(\mathbb{P}(V), \Omega^1(0)) \cong H^0(\mathbb{P}(V), \Omega^0(0)) \cong \mathbb{C}$. For $k = 1$, (4.13a) also shows that $H^0(\mathbb{P}(V), \Omega^1(1)) = 0 = H^1(\mathbb{P}(V), \Omega^1(1))$ while for $k \geq 2$ we find

¹⁹Serre duality (cf. e.g. Griffiths & Harris [40]) asserts that if X is a compact n -dimensional complex manifold and \mathcal{S} an Abelian sheaf on X , then $H^q(X, \Omega_X^p(\mathcal{S})) \cong [H^{n-q}(X, \Omega_X^{n-p}(\mathcal{S}^\vee))]^\vee$.

$H^1(\mathbb{P}(V), \Omega^1(k)) = 0$ since κ is surjective. The rest of $H^0(\mathbb{P}(V), \Omega^1(k))$ then follows from the short exact sequence

$$0 \longrightarrow H^0(\mathbb{P}(V), \Omega^1(k)) \longrightarrow \odot^{k-1} V^\vee \otimes V^\vee \longrightarrow \odot^k V^\vee \longrightarrow 0. \quad (4.14)$$

This concludes the cases $q = 0, 1$.

It remains to find $H^3(\mathbb{P}(V), \Omega^1(k))$. The sequence (4.13b) and Bott's rule show that for $k \geq -2$, $H^3(\mathbb{P}(V), \Omega^1(k)) = 0$ while for $k = -3$, we get $H^3(\mathbb{P}(V), \Omega^1(-3)) \cong V^\vee \otimes \det V$. For $k < -3$, (4.13b) reads as

$$0 \longrightarrow H^3(\mathbb{P}(V), \Omega^1(k)) \longrightarrow \odot^{-k-3} V \otimes \det V \otimes V^\vee \longrightarrow \odot^{-k-4} V \otimes \det V \longrightarrow 0, \quad (4.15)$$

which gives the remaining cases for $H^3(\mathbb{P}(V), \Omega^1(k))$. This completes the proof. \square

Next, we compute the direct image sheaves $\pi_{2*}^q \Omega_{\pi_1}^p(\mathcal{O}_{P^6}(k))$. Using the short-hand notation $\Omega_{\pi_1}^p(k) := \Omega_{\pi_1}^p(\mathcal{O}_{P^6}(k))$, we have the following proposition:

Proposition 4.2. *Let $k_p := 2p + k$. The direct image sheaves $\pi_{2*}^q \Omega_{\pi_1}^p(k)$ are given by*

$$\pi_{2*}^q \Omega_{\pi_1}^0(k) \cong \begin{cases} \odot^{k_0} S & \text{for } q = 0 \text{ \& } k_0 \geq 0, \\ (\odot^{-k_0-4} S^\vee)[1] & \text{for } q = 3 \text{ \& } k_0 \leq -4, \\ 0 & \text{otherwise,} \end{cases} \quad (4.16a)$$

$$\pi_{2*}^q \Omega_{\pi_1}^1(k) \cong \begin{cases} \left(\frac{\odot^{k_1-1} S^\vee \otimes_{\mathcal{O}_{M^6}} S^\vee}{\odot^{k_1} S^\vee} \right)^\vee [1] & \text{for } q = 0 \text{ \& } k_1 \geq 2, \\ [1] & \text{for } q = 1 \text{ \& } k_1 = 0, \\ (\odot^{-k_1-3} S^\vee \otimes_{\mathcal{O}_{M^6}} S)_0[2] & \text{for } q = 3 \text{ \& } k_1 \leq -3, \\ 0 & \text{otherwise,} \end{cases} \quad (4.16b)$$

$$\pi_{2*}^q \Omega_{\pi_1}^2(k) \cong \begin{cases} (\odot^{k_2-3} S \otimes_{\mathcal{O}_{M^6}} S^\vee)_0[1] & \text{for } q = 0 \text{ \& } k_2 \geq 3, \\ [2] & \text{for } q = 2 \text{ \& } k_2 = 0, \\ \left(\frac{\odot^{-k_2-1} S^\vee \otimes_{\mathcal{O}_{M^6}} S^\vee}{\odot^{-k_2} S^\vee} \right)[2] & \text{for } q = 3 \text{ \& } k_2 \leq -2, \\ 0 & \text{otherwise,} \end{cases} \quad (4.16c)$$

and

$$\pi_{2*}^q \Omega_{\pi_1}^3(k) \cong \begin{cases} (\odot^{k_3-4} S)[2] & \text{for } q = 0 \text{ \& } k_3 \geq 4, \\ (\odot^{-k_3} S^\vee)[3] & \text{for } q = 3 \text{ \& } k_3 \leq 0, \\ 0 & \text{otherwise,} \end{cases} \quad (4.16d)$$

where $(\odot^l S^\vee \otimes_{\mathcal{O}_{M^6}} S)_0$ is the totally trace-less part of $\odot^l S^\vee \otimes_{\mathcal{O}_{M^6}} S$ which is

$$(\odot^l S^\vee \otimes_{\mathcal{O}_{M^6}} S)_0 \cong \begin{cases} S & \text{for } l = 0, \\ \frac{\odot^l S^\vee \otimes_{\mathcal{O}_{M^6}} S}{\odot^{l-1} S^\vee} & \text{for } l \geq 1. \end{cases} \quad (4.17)$$

Proof: By definition of direct image sheaves, our task is to compute the cohomology groups $H^q(\pi_2^{-1}(U), \Omega_{\pi_1}^p(k))$ for open sets $U \subset M^6$; see also footnote 14. Notice that it suffices to work with Stein open sets U so that $U' := \pi_2^{-1}(U) \cong U \times \mathbb{P}^3 \subset F^9$ since there are arbitrarily small Stein open sets on M^6 . We could now apply the direct image functor to the short exact sequences of Proposition 4.1 to obtain the direct images. There is, however, a quicker way of computing these.

Consider the case when $p = 0$. It is rather straightforward to see that in this case, we have the identification

$$H^q(U', \Omega_{\pi_1}^0(k)) \cong \{\text{holomorphic functions : } U \rightarrow H^q(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))\}, \quad (4.18)$$

and we can directly apply the results of Lemma 4.1. The other cohomology groups can be characterised analogously. We first recall our discussion of the relative one-forms, $\Omega_{\pi_1}^1(0) = \Omega_{\pi_1}^1$ that led to the sequence (4.8). Let $(x, \lambda) = (x^{AB}, \lambda_A)$ be local coordinates on F^9 , as before. Then the components ω^A of a relative one-form ω are of weight one in λ and obey $\omega^A \lambda_A = 0$. This essentially implies that $\omega^A = \frac{1}{2} \varepsilon^{ABCD} \omega_{BC} \lambda_D$, where $\omega_{AB} = -\omega_{BA}$ depends (holomorphically) on x . Together with our results for the twistor space P^6 presented at the end of Section 3, we may conclude that

$$H^q(U', \Omega_{\pi_1}^1(0)) \cong \{\text{holomorphic functions : } U \rightarrow H^q(\mathbb{P}^3, \Omega^1(2))[1]\}. \quad (4.19)$$

This argument generalises to the remaining cohomology groups $H^q(U', \Omega_{\pi_1}^p)$ for $p = 2, 3$, and we have

$$H^q(U', \Omega_{\pi_1}^p(0)) \cong \{\text{holomorphic functions : } U \rightarrow H^q(\mathbb{P}^3, \Omega^p(2p))[p]\}. \quad (4.20)$$

Therefore, if we let $k_p := 2p + k$, we obtain

$$H^q(U', \Omega_{\pi_1}^p(k)) \cong \{ \text{holomorphic functions} : U \rightarrow H^q(\mathbb{P}^3, \Omega^p(k_p))[p] \} . \quad (4.21)$$

In summary, all the cohomology groups $H^q(\pi_2^{-1}(U), \Omega_{\pi_1}^p(k))$ are characterised in terms of the cohomology groups appearing in Lemma 4.1 for $V = S^\vee$, which yields (4.16). \square

So far, we have computed the direct images of the sheaves $\Omega_{\pi_1}^p(k)$. The resolutions (4.5) and (4.9a) also contain the topological inverse sheaves $\pi_1^{-1}\mathcal{O}_{P^6}$ and $\pi_1^{-1}\mathcal{O}_{P^6}(\mathcal{O}_{P^6}(k))$, respectively. The direct images of these sheaves are computed using spectral sequences. In the following, we shall merely recall a few facts about spectral sequences and we refer to [17] for a more detailed account.

For us, a spectral sequence is basically a sequence of two-dimensional arrays of Abelian groups $E_r = (E_r^{p,q})$ for $r = 1, 2, \dots$ which are labelled by $p, q = 0, 1, 2, \dots$ together with differential operators $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ that obey $d_r \circ d_r = 0$. In addition, the arrays are linked cohomologically from one order to the next. Specifically, we have

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r) := \frac{\ker d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}}{\operatorname{im} d_r : E_r^{p-r, q+r-1} \rightarrow E_r^{p,q}} . \quad (4.22)$$

There also is a well-defined limit of the spectral sequence in terms of the inductive limit

$$E_\infty^{p,q} = \lim_{r \rightarrow \infty} \operatorname{ind} E_r^{p,q} . \quad (4.23)$$

If $U \subset M^6$ is open and $U' := \pi_2^{-1}(U)$, the resolution (4.9a) yields a spectral sequence with initial terms $E_1^{p,q} \cong H^q(U', \Omega_{\pi_1}^p(E))$ and differential operators $d_1 : E_1^{p,q} \rightarrow E_1^{p+1, q}$ induced by $d_{\pi_1} : \Omega_{\pi_1}^p(E) \rightarrow \Omega_{\pi_1}^{p+1}(E)$. This spectral sequence converges to the cohomology group $E_\infty^{p,q} \cong H^{p+q}(U', \pi_1^{-1}\mathcal{O}_{P^6}(E))$, which is mnemonically written as $H^q(U', \Omega_{\pi_1}^p(E)) \Rightarrow H^{p+q}(U', \pi_1^{-1}\mathcal{O}_{P^6}(E))$. Altogether, we have the following proposition:

Proposition 4.3. *Let U be an open set in M^6 and let $U' := \pi_2^{-1}(U) \subset F^9$. Then there is a spectral sequence*

$$E_1^{p,q} \cong H^q(U', \Omega_{\pi_1}^p(k)) \Longrightarrow H^{p+q}(U', \pi_1^{-1}\mathcal{O}_{P^6}(k)) , \quad (4.24)$$

where the differential operators $d_1 : E_1^{p,q} \rightarrow E_1^{p+1, q}$ are induced by the relative exterior derivative $d_{\pi_1} : \Omega_{\pi_1}^p(k) \rightarrow \Omega_{\pi_1}^{p+1}(k)$.

Hence, we have an explicit way of computing $H^q(U', \pi_1^{-1}\mathcal{O}_{P^6}(k))$ in terms of the cohomology groups $H^q(U', \Omega_{\pi_1}^p(k))$.

Cohomology groups of topological inverse sheaves. The final ingredient we need is a result due to Buchdahl [41]. Above we have computed the direct images of sheaves on the correspondence space F^9 along the fibration $\pi_2 : F^9 \rightarrow M^6$ to obtain certain sheaves on space-time M^6 . In the Penrose transform, these sheaves on F^9 originate from sheaves on twistor space. To connect the cohomology groups of both kinds of sheaves, we can use the following proposition, which we recall for the reader's convenience:

Proposition 4.4. (*Buchdahl [41]*) *Let X and Y be complex manifolds and $\pi : X \rightarrow Y$ a surjective holomorphic mapping of maximal rank with connected fibres. Furthermore, let \mathcal{S} be an Abelian sheaf on Y . If there is an $n_0 > 0$ such that $H^q(\pi^{-1}(p), \mathbb{C}) = 0$ for $q = 1, \dots, n_0$ and for all $p \in Y$, then*

$$\pi^* : H^q(Y, \mathcal{S}) \rightarrow H^q(X, \pi^{-1}\mathcal{S}) \quad (4.25)$$

is an isomorphism for $q = 0, \dots, n_0$ and a monomorphism for $q = n_0 + 1$.

The requirements of this proposition for the projection $\pi_1 : F^9 \rightarrow P^6$ are always satisfied in our setting. Because we always work with convex subsets $U \subset M^6$, we always have the isomorphism $H^q(U', \mathcal{S}) \cong H^q(\hat{U}, \pi_1^{-1}\mathcal{S})$, where $U' := \pi_2^{-1}(U) \subset F^9$ and $\hat{U} := \pi_1(\pi_2^{-1}(U)) \subset P^6$. In a compactified version of the twistor correspondence, one has to supplement Theorem 4.1 by the above requirements.

4.2. Proof

We are now ready to prove Theorem 4.1. We shall first proof the case $h > 0$, that is $-2h - 4 < -4$, and then come to the case $h = 0$, which is slightly more complicated.

Case $h > 0$. Recall that sections ψ of the sheaf \mathcal{Z}_s defined in (2.9) obey the free field equation

$$\partial^{AB}\psi_{BA_1\dots A_{2s-1}} = 0. \quad (4.26)$$

We thus have to prove that for $s = h$,

$$\mathcal{P} : H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)) \rightarrow H^0(U, \mathcal{Z}_s) \quad (4.27)$$

is an isomorphism. We already know from Proposition 4.4 that

$$H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)) \cong H^3(U', \pi_1^{-1}\mathcal{O}_{\hat{U}}(-2h-4)), \quad (4.28)$$

which reduces (4.27) to

$$H^3(U', \pi_1^{-1}\mathcal{O}_{\hat{U}}(-2h-4)) \cong H^0(U, \mathcal{Z}_h). \quad (4.29)$$

Firstly, we notice that there is a particular spectral sequence, the *Leray spectral sequence* $L_r = (L_r^{p,q})$, which gives²⁰

$$L_2^{p,q} \cong H^p(U, \pi_{2*}^q \Omega_{\pi_1}^l(-2h-4)) \implies H^{p+q}(U', \Omega_{\pi_1}^l(-2h-4)) . \quad (4.30)$$

For fixed l , Proposition 4.2 for $h > 0$ tells us that $\pi_{2*}^q \Omega_{\pi_1}^l(-2h-4) = 0$ if $q \neq 3$. Thus, the Leray spectral sequence $L_r^{p,q}$ is degenerate at the second level. Therefore, we have

$$L_\infty^{p,q} \cong L_2^{p,q} \quad \text{for } p, q \geq 0 , \quad (4.31)$$

cf. (4.22). Recall that if a spectral sequence $(E_r^{p,q})$ has the property that for some r_0 , $E_{r_0}^{p,q} = 0$ for $q \neq q_0$, then $E_{r_0}^{p,q_0} \cong H^{p+q_0}$. This property together with (4.30) then imply

$$H^p(U', \Omega_{\pi_1}^l(-2h-4)) \cong \begin{cases} H^{p-3}(U, \pi_{2*}^3 \Omega_{\pi_1}^l(-2h-4)) & \text{for } p \geq 3 , \\ 0 & \text{for } p < 3 . \end{cases} \quad (4.32)$$

Secondly, Proposition 4.3 yields another spectral sequence $E_r = (E_r^{p,q})$ with

$$E_1^{p,q} \cong H^q(U', \Omega_{\pi_1}^p(-2h-4)) \implies H^{p+q}(U', \pi_1^{-1} \mathcal{O}_{P^6}(-2h-4)) . \quad (4.33)$$

Explicitly, the $r = 1$ array in this sequence reads as ($k = -2h - 4$)

$$\begin{array}{ccccccc} H^0(U', \Omega_{\pi_1}^0(k)) & \xrightarrow{d_{\pi_1}} & H^0(U', \Omega_{\pi_1}^1(k)) & \xrightarrow{d_{\pi_1}} & H^0(U', \Omega_{\pi_1}^2(k)) & \xrightarrow{d_{\pi_1}} & H^0(U', \Omega_{\pi_1}^3(k)) \\ H^1(U', \Omega_{\pi_1}^0(k)) & \xrightarrow{d_{\pi_1}} & H^1(U', \Omega_{\pi_1}^1(k)) & \xrightarrow{d_{\pi_1}} & H^1(U', \Omega_{\pi_1}^2(k)) & \xrightarrow{d_{\pi_1}} & H^1(U', \Omega_{\pi_1}^3(k)) \\ H^2(U', \Omega_{\pi_1}^0(k)) & \xrightarrow{d_{\pi_1}} & H^2(U', \Omega_{\pi_1}^1(k)) & \xrightarrow{d_{\pi_1}} & H^2(U', \Omega_{\pi_1}^2(k)) & \xrightarrow{d_{\pi_1}} & H^2(U', \Omega_{\pi_1}^3(k)) \\ H^3(U', \Omega_{\pi_1}^0(k)) & \xrightarrow{d_{\pi_1}} & H^3(U', \Omega_{\pi_1}^1(k)) & \xrightarrow{d_{\pi_1}} & H^3(U', \Omega_{\pi_1}^2(k)) & \xrightarrow{d_{\pi_1}} & H^3(U', \Omega_{\pi_1}^3(k)) \\ H^4(U', \Omega_{\pi_1}^0(k)) & \xrightarrow{d_{\pi_1}} & H^4(U', \Omega_{\pi_1}^1(k)) & \xrightarrow{d_{\pi_1}} & H^4(U', \Omega_{\pi_1}^2(k)) & \xrightarrow{d_{\pi_1}} & H^4(U', \Omega_{\pi_1}^3(k)) \\ \vdots & & \vdots & & \vdots & & \vdots \end{array} \quad (4.34)$$

²⁰In general, if \mathcal{S} is an Abelian sheaf on X and $\pi : X \rightarrow Y$, the Leray spectral sequence $L_r = (L_r^{p,q})$ relates the cohomology of \mathcal{S} to that of its direct images (see e.g. Godement [43]) according to $L_2^{p,q} \cong H^p(Y, \pi_*^q \mathcal{S}) \Rightarrow H^{p+q}(X, \mathcal{S})$.

We may now replace these cohomology groups by $H^q(U', \Omega_{\pi_1}^p(k))$ using (4.32) to obtain

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
0 & & 0 & & 0 \\
0 & & 0 & & 0 \\
H^0(U, \pi_{2*}^3 \Omega_{\pi_1}^0(k)) & \longrightarrow & H^0(U, \pi_{2*}^3 \Omega_{\pi_1}^1(k)) & \longrightarrow & \cdots \longrightarrow H^0(U, \pi_{2*}^3 \Omega_{\pi_1}^3(k)) \\
H^1(U, \pi_{2*}^3 \Omega_{\pi_1}^0(k)) & \longrightarrow & H^1(U, \pi_{2*}^3 \Omega_{\pi_1}^1(k)) & \longrightarrow & \cdots \longrightarrow H^1(U, \pi_{2*}^3 \Omega_{\pi_1}^3(k)) \\
H^2(U, \pi_{2*}^3 \Omega_{\pi_1}^0(k)) & \longrightarrow & H^2(U, \pi_{2*}^3 \Omega_{\pi_1}^1(k)) & \longrightarrow & \cdots \longrightarrow H^2(U, \pi_{2*}^3 \Omega_{\pi_1}^3(k)) \\
H^3(U, \pi_{2*}^3 \Omega_{\pi_1}^0(k)) & \longrightarrow & H^3(U, \pi_{2*}^3 \Omega_{\pi_1}^1(k)) & \longrightarrow & \cdots \longrightarrow H^3(U, \pi_{2*}^3 \Omega_{\pi_1}^3(k)) \\
\vdots & & \vdots & & \vdots
\end{array} \tag{4.35}$$

This diagram together with (4.22) then yield the following identification:

$$E_2^{0,3} \cong \ker \{ H^0(U, \pi_{2*}^3 \Omega_{\pi_1}^0(-2h-4)) \rightarrow H^0(U, \pi_{2*}^3 \Omega_{\pi_1}^1(-2h-4)) \}. \tag{4.36}$$

Furthermore, all $E_r^{p,q} = 0$ for $p+q=3$ with $q \neq 3$, and $E_2^{0,3} \cong E_3^{0,3} \cong \cdots \cong E_\infty^{0,3}$. From Proposition 4.2, it follows that $\pi_{2*}^3 \Omega_{\pi_1}^0(-2h-4) \cong (\odot^{2h} S^\vee)[1]$ and $\pi_{2*}^3 \Omega_{\pi_1}^1(-2h-4) \cong (\odot^{2h-1} S^\vee \otimes_{\mathcal{O}_U} S)_0[2]$. In addition, the relative exterior derivative $d_{\pi_1} : H^3(U', \Omega_{\pi_1}^0(k)) \rightarrow H^3(U', \Omega_{\pi_1}^1(k))$ induces the differential operator

$$\partial^{AB} : H^0(U, \pi_{2*}^3 \Omega_{\pi_1}^0(-2h-4)) \rightarrow H^0(U, \pi_{2*}^3 \Omega_{\pi_1}^1(-2h-4)). \tag{4.37}$$

In summary, from (4.28) and (4.33) we may therefore conclude that

$$H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)) \cong H^3(U', \pi_1^{-1} \mathcal{O}_{\hat{U}}(-2h-4)) \cong E_2^{0,3} \cong H^0(U, \mathcal{Z}_h). \tag{4.38}$$

Case $h=0$. The proof for $h=0$ is similar to the one presented above albeit somewhat more difficult. Firstly, we shall be dealing with a second-order partial differential operator and secondly, on a more technical level, the appropriate spectral sequence will degenerate differently.

Recall that \mathcal{Z}_0 is the sheaf of solutions to the Klein–Gordon equation. That is, its sections describe scalar fields on space-time forming the trivial representation $(\mathbf{1}, \mathbf{1})$ under the little group. We wish to prove that

$$\mathcal{P} : H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-4)) \rightarrow H^0(U, \mathcal{Z}_0) \tag{4.39}$$

is an isomorphism. Again, by virtue of Proposition 4.4, we only need to show that

$$H^3(U', \pi_1^{-1} \mathcal{O}_{\hat{U}}(-4)) \cong H^0(U, \mathcal{Z}_0). \tag{4.40}$$

From Proposition 4.2, we see that

$$\pi_{2*}^q \Omega_{\pi_1}^l(-4) \cong \begin{cases} [1] & \text{for } (q, l) = (3, 0) , \\ [2] & \text{for } (q, l) = (2, 2) , \\ 0 & \text{otherwise .} \end{cases} \quad (4.41)$$

When $(q, l) = (3, 0)$, the corresponding Leray spectral sequence (4.30) yields

$$H^p(U', \Omega_{\pi_1}^0(-4)) \cong \begin{cases} H^{p-3}(U, \pi_{2*}^3 \Omega_{\pi_1}^0(-4)) \cong H^{p-3}(U, [1]) & \text{for } p \geq 3 , \\ 0 & \text{for } p < 3 . \end{cases} \quad (4.42)$$

Moreover, with (4.41) the Leray spectral sequence (4.30) also gives

$$H^p(U, \pi_{2*}^q \Omega_{\pi_1}^l(-4)) = 0 \quad \text{for } p, q \geq 0 \quad \text{and } l = 1, 3 . \quad (4.43)$$

When $(q, l) = (2, 2)$, we derive

$$H^p(U', \Omega_{\pi_1}^2(-4)) \cong \begin{cases} H^{p-2}(U, \pi_{2*}^2 \Omega_{\pi_1}^2(-4)) \cong H^{p-2}(U, [2]) & \text{for } p \geq 2 , \\ 0 & \text{for } p < 2 . \end{cases} \quad (4.44)$$

Next, the $r = 1$ part of the spectral sequence (4.33) for $h = 0$ is given by

$$\begin{array}{ccccccc} H^0(U', \Omega_{\pi_1}^0(-4)) & \xrightarrow{d_{\pi_1}} & H^0(U', \Omega_{\pi_1}^1(-4)) & \xrightarrow{d_{\pi_1}} & H^0(U', \Omega_{\pi_1}^2(-4)) & \xrightarrow{d_{\pi_1}} & H^0(U', \Omega_{\pi_1}^3(-4)) \\ H^1(U', \Omega_{\pi_1}^0(-4)) & \xrightarrow{d_{\pi_1}} & H^1(U', \Omega_{\pi_1}^1(-4)) & \xrightarrow{d_{\pi_1}} & H^1(U', \Omega_{\pi_1}^2(-4)) & \xrightarrow{d_{\pi_1}} & H^1(U', \Omega_{\pi_1}^3(-4)) \\ H^2(U', \Omega_{\pi_1}^0(-4)) & \xrightarrow{d_{\pi_1}} & H^2(U', \Omega_{\pi_1}^1(-4)) & \xrightarrow{d_{\pi_1}} & H^2(U', \Omega_{\pi_1}^2(-4)) & \xrightarrow{d_{\pi_1}} & H^2(U', \Omega_{\pi_1}^3(-4)) \\ H^3(U', \Omega_{\pi_1}^0(-4)) & \xrightarrow{d_{\pi_1}} & H^3(U', \Omega_{\pi_1}^1(-4)) & \xrightarrow{d_{\pi_1}} & H^3(U', \Omega_{\pi_1}^2(-4)) & \xrightarrow{d_{\pi_1}} & H^3(U', \Omega_{\pi_1}^3(-4)) \\ H^4(U', \Omega_{\pi_1}^0(-4)) & \xrightarrow{d_{\pi_1}} & H^4(U', \Omega_{\pi_1}^1(-4)) & \xrightarrow{d_{\pi_1}} & H^4(U', \Omega_{\pi_1}^2(-4)) & \xrightarrow{d_{\pi_1}} & H^4(U', \Omega_{\pi_1}^3(-4)) \\ \vdots & & \vdots & & \vdots & & \vdots \end{array} \quad (4.45)$$

Our above calculations show that the second and third columns of this diagram are zero, while the first and fourth ones are non-zero. Hence, the differential operator d_1 on $E_1^{p,q}$ vanishes identically and therefore, we have the identification $E_1^{p,q} \cong E_2^{p,q}$. Substituting

(4.42)–(4.44) into this diagram, we eventually find

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & H^0(U, [2]) & \longrightarrow & 0 \\
H^0(U, [1]) & \longrightarrow & 0 & \longrightarrow & H^1(U, [2]) & \longrightarrow & 0 \\
H^1(U, [1]) & \longrightarrow & 0 & \longrightarrow & H^2(U, [2]) & \longrightarrow & 0 \\
H^2(U, [1]) & \longrightarrow & 0 & \longrightarrow & H^3(U, [2]) & \longrightarrow & 0 \\
H^3(U, [1]) & \longrightarrow & 0 & \longrightarrow & H^4(U, [2]) & \longrightarrow & 0 \\
\vdots & & & & \vdots & &
\end{array} \tag{4.46}$$

Furthermore, the differential operator d_2 on $E_2^{0,3}$ maps $E_2^{0,3}$ to $E_2^{2,2}$ and since $E_1^{p,q} \cong E_2^{p,q}$ and thus, $E_2^{0,3} \cong H^0(U, [1])$ and $E_2^{2,2} \cong H^0(U, [2])$, respectively, we have a map $\square : H^0(U, [1]) \rightarrow H^0(U, [2])$ which is induced by d_2 . One can see that this map is a composition of first-order differential operators and it is indeed the one we defined in (2.9).

Finally, we note that

$$E_3^{0,3} \cong \ker \{ \square : H^0(U, [1]) \rightarrow H^0(U, [2]) \} , \tag{4.47}$$

together with $E_r^{p,q} = 0$ for $p + q = 3$ with $q \neq 3$, and also $E_3^{0,3} \cong \dots \cong E_\infty^{0,3}$. Altogether,

$$H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-4)) \cong H^3(U', \pi_1^{-1} \mathcal{O}_{\hat{U}}(-4)) \cong E_3^{0,3} \cong H^0(U, \mathcal{Z}_0) , \tag{4.48}$$

which completes the proof for $h = 0$.

4.3. Integral formulæ and thickenings

Similarly to four dimensions [3], we can write down certain contour integral formulæ yielding solutions to the zero-rest-mass field equations in six dimensions. As already indicated, such formulæ appeared first in works by Hughston [27] and more recently, they were discussed by Berkovits & Cherkis in [29] and Chern in [30].

Let us choose a Stein covering $\hat{\mathfrak{U}} = \{\hat{U}_{[a]}\}$ of \hat{U} . We shall make use of the abbreviations $\hat{U}_{[ab]} := \hat{U}_{[a]} \cap \hat{U}_{[b]}$, $\hat{U}_{[abc]} := \hat{U}_{[a]} \cap \hat{U}_{[b]} \cap \hat{U}_{[c]}$, etc. The simplest choice for $\hat{\mathfrak{U}}$ is a lift of the standard cover of \mathbb{P}^3 to \hat{U} requiring four patches $\hat{U}_{[a]}$, $a = 1, \dots, 4$. In this case, there is only one quadruple overlap of four patches, and a holomorphic function $\hat{f}_{-2h-4} = \hat{f}_{-2h-4}(z, \lambda)$ on $\hat{U}_{[1234]} \subset \hat{U}$ of homogeneity $-2h - 4$ represents an element of $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h - 4))$. For simplicity, we shall assume a Čech cocycle \hat{f}_{-2h-4} of this form in the following. Note

that this is not the most general way of representing elements of $H^3(\hat{U}, \mathcal{O}_{\hat{U}})$. This, however, requires merely a technical extension of our discussion below using branched contour integrals, cf. [16].

In the following, we shall construct zero-rest-mass fields $\psi \in H^0(U, \mathcal{Z}_s) \subset H^0(U, \odot^{2s} S^\vee)$, $s \in \frac{1}{2}\mathbb{N}_0$. That is, ψ forms the representation $(2\mathbf{s} + \mathbf{1}, \mathbf{1})$ of the little group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. We start from a Čech cocycle \hat{f}_{-2h-4} with $h = s \geq 0$, which we restrict to $\hat{x} \cong \mathbb{P}^3$ to obtain $\hat{f}_{-2h-4} = \hat{f}_{-2h-4}(x \cdot \lambda, \lambda)$ on the intersection $\hat{U}_{[1234]} \cap \hat{x}$. Using the holomorphic $\mathrm{SL}(4, \mathbb{C})$ -invariant measure on \mathbb{P}^3 given by

$$\Omega^{(3,0)} := \frac{1}{4!} \varepsilon^{ABCD} \lambda_A d\lambda_B \wedge d\lambda_C \wedge d\lambda_D, \quad (4.49)$$

we can write down the contour integral

$$\psi_{A_1 \dots A_{2s}}(x) = \oint_{\mathcal{C}} \Omega^{(3,0)} \lambda_{A_1} \cdots \lambda_{A_{2s}} \hat{f}_{-2s-4}(x \cdot \lambda, \lambda), \quad (4.50)$$

where the contour \mathcal{C} is topologically a three-torus contained in $\hat{U}_{[1234]}$. Clearly

$$\partial^{AB} \psi_{BA_1 \dots A_{2s-1}} = 0 \quad \text{for } s > 0 \quad \text{and} \quad \square \psi = 0 \quad \text{for } s = 0, \quad (4.51)$$

as follows from straightforward differentiation under the integral.

Notice that there is another way of obtaining zero-rest-mass fields by choosing $h = -s$. This case is less straightforward since the integral formulæ given in [29, 30] implicitly involve so-called *thickenings* (also known as *infinitesimal neighbourhoods*) of the twistor space P^6 into its ambient space $\mathbb{P}^7_\circ \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4$.²¹ Notice that thickenings of manifolds occur in various contexts within twistor geometry. The most prominent examples appear in the twistor descriptions of Yang–Mills theory and Einstein gravity [8–13] in four space-time dimensions (see also Section 6.2). To thicken our twistor space P^6 , consider $\mathcal{O}_{\mathbb{P}^7_\circ}$, the sheaf of holomorphic functions on \mathbb{P}^7_\circ , and \mathcal{I} , the ideal subsheaf of $\mathcal{O}_{\mathbb{P}^7_\circ}$ consisting of those functions that vanish on $P^6 \hookrightarrow \mathbb{P}^7_\circ$. The ℓ -th order thickening (or ℓ -th infinitesimal neighbourhood) of P^6 inside \mathbb{P}^7_\circ is denoted by $P^6_{[\ell]}$ and it is defined by

$$P^6_{[\ell]} := (P^6, \mathcal{O}_{\mathbb{P}^7_\circ} / \mathcal{I}^{\ell+1}). \quad (4.52)$$

Notice that we recover the twistor space as the zeroth order thickening, i.e. $P^6_{[0]} = P^6$. Moreover, a cover of P^6 will also form a cover of $P^6_{[\ell]}$. The spaces $P^6_{[\ell]}$ can be thought of as the jets of the embedding of P^6 into the larger manifold \mathbb{P}^7_\circ . In local coordinates (z^A, λ_A) on \mathbb{P}^7_\circ , we have $(z^A \lambda_A)^{i+1} = 0$ for $i \geq \ell$ but $(z^A \lambda_A)^i \neq 0$ for $0 < i \leq \ell$ on $P^6_{[\ell]}$. This implies that on the first order thickening $P^6_{[1]}$, the four vector fields $\frac{\partial}{\partial z^A}$ are linearly independent

²¹Recall that P^6 is a hypersurface in \mathbb{P}^7_\circ as follows from (3.9a).

and act freely on functions on $P_{[1]}^6$. Differential operators of order ℓ constructed out of these four vector fields act freely on functions on $P_{[\ell]}^6$. As we shall see momentarily, this fact is the essential ingredient for writing down a contour integral leading to zero-rest-mass fields.

Proceeding analogously to four dimensions, we shall now construct a second contour integral by replacing λ_A in (4.50) by the derivatives $\frac{\partial}{\partial z^A}$ and adjusting the homogeneity of \hat{f}_{-2h-4} accordingly. The resulting $-2h$ derivatives in the contour integral should act freely, and therefore we have to consider $P_{[-2h]}^6$. Let $\hat{f}_{-2h-4}^{[-2h]} = \hat{f}_{-2h-4}^{[-2h]}(z, \lambda)$ be a representative of the cohomology group $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ defined on $\hat{U}_{[1234]} \subset \hat{U} \subset P_{[-2h]}^6$. It is expanded as

$$\hat{f}_{-2h-4}^{[-2h]}(z, \lambda) = \hat{g}(\lambda) + \sum_{l \geq 1} \frac{1}{l!} z^{A_1} \cdots z^{A_l} \hat{g}_{A_1 \dots A_l}(\lambda), \quad (4.53)$$

where the coefficients $\hat{g}_{A_1 \dots A_l}$ for $l \leq -2h$ are uniquely defined for $0 < l \leq -2h$. We may rewrite the above expansion as

$$\hat{f}_{-2h-4}^{[-2h]}(z, \lambda) = \frac{1}{(-2h)!} z^{A_1} \cdots z^{A_{-2h}} \hat{f}_{A_1 \dots A_{-2h}}(z, \lambda) + \cdots, \quad (4.54)$$

where the ellipsis denotes terms that contain at most $-2h-1$ factors of z^A . As the coefficients $\hat{f}_{A_1 \dots A_{-2h}}$ are uniquely fixed, they can be extracted from $\hat{f}_{-2h-4}^{[-2h]}$. On the intersection $\hat{U}_{[1234]} \cap \hat{x}$, we may write

$$\hat{f}_{A_1 \dots A_{-2h}}(x \cdot \lambda, \lambda) = \left. \frac{\partial}{\partial z^{A_1}} \cdots \frac{\partial}{\partial z^{A_{-2h}}} \hat{f}_{-2h-4}^{[-2h]}(z, \lambda) \right|_{z=x \cdot \lambda}. \quad (4.55)$$

The latter relation can then be used to construct the contour integral formula

$$\begin{aligned} \psi_{A_1 \dots A_{2s}}(x) &= \oint_{\mathcal{C}} \Omega^{(3,0)} \hat{f}_{A_1 \dots A_{2s}}(x \cdot \lambda, \lambda) \\ &= \oint_{\mathcal{C}} \Omega^{(3,0)} \frac{\partial}{\partial z^{A_1}} \cdots \frac{\partial}{\partial z^{A_{2s}}} \hat{f}_{2s-4}^{[2s]}(z, \lambda) \Big|_{z=x \cdot \lambda}, \end{aligned} \quad (4.56)$$

where the contour is again a three-torus. By differentiation under the integral, one may check that this is indeed a zero-rest-mass field, i.e.

$$\partial^{AB} \psi_{BA_1 \dots A_{2s-1}} = 0, \quad (4.57)$$

since $\frac{\partial}{\partial x^{AB}} = \lambda_{[A} \frac{\partial}{\partial z^{B]}}$ under the integral.

More generally, we can write down the following contour integral, which interpolates between the above two formulæ (4.50) and (4.56):

$$\psi_{A_1 \dots A_{2s}}(x) = \oint_{\mathcal{C}} \Omega^{(3,0)} \lambda_{(A_1} \cdots \lambda_{A_{h+s}} \frac{\partial}{\partial z^{A_{h+s+1}}} \cdots \frac{\partial}{\partial z^{A_{2s}}} \hat{f}_{-2h-4}^{[s-h]}(z, \lambda) \Big|_{z=x \cdot \lambda}. \quad (4.58)$$

Here, $h = -s, \dots, s$ and the indices A_1, \dots, A_{2s} are symmetrised in the integrand. Again, it is straightforward to check that these fields satisfy the field equation $\partial^{AB}\psi_{BA_1\dots A_{2s-1}} = 0$.

It is usually easier to re-interpret thickenings of a twistor space as a restriction of a supermanifold that has the twistor space as its body, see e.g. [44]. Correspondingly, we expect it to be easier to prove a supersymmetric version of the above extension of Theorem 4.1. This, however, is beyond the scope of this work. Let us therefore rather treat the case $h < 0$ via Penrose–Ward transforms on P^6 .

5. Penrose–Ward transform in six dimensions

In the previous section, we have seen how the Penrose transform relates $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ for $h \geq 0$ to spinor fields on space-time. Subject of this section is the discussion for $h < 0$. As already indicated, one way to tackle this case uses normal expansions and the machinery of spectral sequences in the spirit of the four-dimensional procedure, cf. [45, 32]. Here, however, we shall follow a different route via Riemann–Hilbert problems and gauge potentials instead, cf. [46, 32]. That is, we generalise the idea of Penrose–Ward transforms [5] to the twistor space P^6 .

Spinor fields from $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$, $h = -2$. We begin our considerations with elements of the cohomology group²² $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(0)) \equiv H^3(\hat{U}, \mathcal{O}_{\hat{U}})$. Such elements encode two-gerbes subject to certain triviality conditions,²³ as we shall see momentarily.

An alternative characterisation of $H^3(\hat{U}, \mathcal{O}_{\hat{U}})$ can be derived from the exponential sheaf sequence on \hat{U} ,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\hat{U}} \xrightarrow{\exp} \mathcal{O}_{\hat{U}}^* \longrightarrow 0. \quad (5.1)$$

Here, $\mathcal{O}_{\hat{U}}^*$ is the sheaf of non-vanishing holomorphic functions on \hat{U} and $\exp : \mathcal{O}_{\hat{U}} \rightarrow \mathcal{O}_{\hat{U}}^*$ is the exponential map $\exp(f) := e^{2\pi\sqrt{-1}f}$. The induced long exact sequence of cohomology groups on \hat{U} then reads as

$$\dots \longrightarrow H^3(\hat{U}, \mathbb{Z}) \longrightarrow H^3(\hat{U}, \mathcal{O}_{\hat{U}}) \longrightarrow H^3(\hat{U}, \mathcal{O}_{\hat{U}}^*) \xrightarrow{\text{cl}} H^4(\hat{U}, \mathbb{Z}) \longrightarrow \dots \quad (5.2)$$

One can check²⁴ that $H^3(\hat{U}, \mathbb{Z}) = 0$, and therefore

$$0 \longrightarrow H^3(\hat{U}, \mathcal{O}_{\hat{U}}) \longrightarrow H^3(\hat{U}, \mathcal{O}_{\hat{U}}^*) \xrightarrow{\text{cl}} H^4(\hat{U}, \mathbb{Z}). \quad (5.3)$$

²²We shall again consider open subsets $U \subset M^6$ of space-time together with the corresponding subsets $U' = \pi_2^{-1}(U) \subset F^9$ of the correspondence space and $\hat{U} = \pi_1(\pi_2^{-1}(U)) \subset P^6$ of twistor space.

²³Some basic facts on n -gerbes are collected in Appendix B.

²⁴There are no odd-dimensional cells in the cell decomposition of \hat{U} .

Notice that $H^3(\hat{U}, \mathcal{O}_{\hat{U}}^*)$ is the moduli space of holomorphic two-gerbes on \hat{U} , and cl is the map to their characteristic classes²⁵ in $H^4(\hat{U}, \mathbb{Z})$. The short exact sequence (5.3) allows us to identify $H^3(\hat{U}, \mathcal{O}_{\hat{U}})$ with the kernel of the map cl ,

$$H^3(\hat{U}, \mathcal{O}_{\hat{U}}) \cong \{ \hat{\Gamma} \in H^3(\hat{U}, \mathcal{O}_{\hat{U}}^*) \mid \text{cl}(\hat{\Gamma}) = 0 \} . \quad (5.4)$$

That is, elements of $H^3(\hat{U}, \mathcal{O}_{\hat{U}})$ describe holomorphic two-gerbes which become holomorphically trivial when restricted to any $\hat{x} = \pi_1(\pi_2^{-1}(x)) \hookrightarrow \hat{U}$ for $x \in U$ (remember that $\hat{x} \cong \mathbb{P}^3$).

To make our constructions explicit, let us choose a Stein cover $\hat{\mathcal{U}} = \{\hat{U}_{[a]}\}$ of \hat{U} and a (smooth) partition of unity $\hat{\theta} = \{\hat{\theta}_{[a]}\}$ subordinate to $\hat{\mathcal{U}}$. As before, we shall write $\hat{U}_{[ab]} := \hat{U}_{[a]} \cap \hat{U}_{[b]}$, $\hat{U}_{[abc]} := \hat{U}_{[a]} \cap \hat{U}_{[b]} \cap \hat{U}_{[c]}$, etc.

Consider a holomorphic two-gerbe $\hat{\Gamma}$ on \hat{U} , which is given by a Čech cocycle $\hat{f} = \{\hat{f}_{[abcd]}\} \in H^3(\hat{U}, \mathcal{O}_{\hat{U}})$. Recall that the Dolbeault isomorphism allows us to identify the Čech cohomology group $H^3(\hat{U}, \mathcal{O}_{\hat{U}})$ with the Dolbeault cohomology group $H_{\bar{\partial}}^{(0,3)}(\hat{U})$ of $\bar{\partial}$ -closed $(0,3)$ -forms on \hat{U} .²⁶ Explicitly, this is done using the partition of unity $\hat{\theta} = \{\hat{\theta}_{[a]}\}$: We may introduce a smooth Čech two-cochain \hat{s} by setting

$$\hat{s}_{[abc]} := \sum_d \hat{f}_{[abcd]} \hat{\theta}_{[d]} \quad \text{on} \quad \hat{U}_{[abc]} . \quad (5.5)$$

This cochain gives rise to a smooth splitting of \hat{f} ,

$$\hat{f}_{[abcd]} = \hat{s}_{[abc]} - \hat{s}_{[bcd]} + \hat{s}_{[cda]} - \hat{s}_{[dab]} \quad \text{on} \quad \hat{U}_{[abcd]} . \quad (5.6)$$

From this splitting, we can now define $(0, q)$ -forms with $q = 1, 2, 3$ by

$$\begin{aligned} \hat{A}_{[abc]} &:= \bar{\partial} \hat{s}_{[abc]} = \sum_d \hat{f}_{[abcd]} \bar{\partial} \hat{\theta}_{[d]} \quad \text{on} \quad \hat{U}_{[abc]} , \\ \hat{B}_{[ab]} &:= \sum_{c,d} \hat{f}_{[abcd]} \bar{\partial} \hat{\theta}_{[c]} \wedge \bar{\partial} \hat{\theta}_{[d]} \quad \text{on} \quad \hat{U}_{[ab]} , \\ \hat{C}_{[a]} &:= \sum_{b,c,d} \hat{f}_{[abcd]} \bar{\partial} \hat{\theta}_{[b]} \wedge \bar{\partial} \hat{\theta}_{[c]} \wedge \bar{\partial} \hat{\theta}_{[d]} \quad \text{on} \quad \hat{U}_{[a]} . \end{aligned} \quad (5.7)$$

These $(0, q)$ -forms define a so-called *holomorphic connective structure* on $\hat{\Gamma}$. Clearly, they are all $\bar{\partial}$ -closed on the respective intersections of the coordinate patches $\hat{U}_{[a]}$. Furthermore, the $(0, 3)$ -forms $\hat{C}_{[a]}$ yield a globally defined, i.e. defined on all of \hat{U} , $(0, 3)$ -form $\hat{C}^{(0,3)}$ with $\hat{C}_{[a]} = \hat{C}^{(0,3)}|_{\hat{U}_{[a]}}$ since $\hat{C}_{[a]} = \sum_{b,c,d} \hat{s}_{[bcd]} \bar{\partial} \hat{\theta}_{[b]} \wedge \bar{\partial} \hat{\theta}_{[c]} \wedge \bar{\partial} \hat{\theta}_{[d]}$, which is $\bar{\partial}$ -closed. This is the

²⁵As explained in Appendix B, this map is analogous to the map $H^1(\hat{U}, \mathcal{O}_{\hat{U}}^*) \xrightarrow{c_1} H^2(\hat{U}, \mathbb{Z})$, taking transition functions of holomorphic line bundles to their first Chern classes.

²⁶Strictly speaking, they take values in some Abelian Lie algebra \mathfrak{g} .

desired Dolbeault representative. We therefore have $\hat{G}^{(0,4)} := \bar{\partial}\hat{C}^{(0,3)} = 0$, which is the two-gerbe analogue of the equations of motion of (Abelian) holomorphic Chern–Simons theory for holomorphic vector bundles. Here, $\hat{G}^{(0,4)}$ is understood as the $(0,4)$ -part of the four-form curvature.

Because $\hat{\Gamma}$ is holomorphically trivial on any $\hat{x} \hookrightarrow \hat{U}$, we have a holomorphic splitting of \hat{f} on \hat{x} ,

$$\hat{f}_{[abcd]} = \hat{h}_{[abc]} - \hat{h}_{[bcd]} + \hat{h}_{[cda]} - \hat{h}_{[dab]} \quad \text{on} \quad \hat{U}_{[abcd]} \cap \hat{x}. \quad (5.8)$$

Here, the Čech two-cochain $\hat{h} = \{\hat{h}_{[abc]}\}$ is holomorphic, i.e. $\hat{h} = \hat{h}(x, \lambda)$ depends holomorphically on (x, λ) . Finding such a splitting is known as *Riemann–Hilbert problem*. Notice that such splittings are not unique, as we can always shift $\hat{h}_{[abc]} \mapsto \hat{h}_{[abc]} + \hat{\varphi}_{[ab]} + \hat{\varphi}_{[bc]} + \hat{\varphi}_{[ca]}$ for holomorphic $\hat{\varphi} = \{\hat{\varphi}_{[ab]}\}$.

Consider now the cover $\mathcal{U}' := \{U'_{[a]}\}$ of $U' \subset F^9$ which is induced by $\hat{\mathcal{U}}$, i.e. $U'_{[a]} = \pi_1^{-1}(\hat{U}_{[a]})$, and a (smooth) partition of unity $\theta' = \{\theta'_{[a]}\}$ subordinate to \mathcal{U}' . The pull-back $f' = \pi_1^* \hat{f}$ defines the pull-back two-gerbe $\Gamma' = \pi_1^* \hat{\Gamma}$ on U' . This two-gerbe is holomorphically trivial on U' (since $\hat{\Gamma}$ was so on any \hat{x}), and we therefore have a holomorphic splitting

$$f'_{[abcd]} = \pi_1^* \hat{f}_{[abcd]} = h'_{[abc]} - h'_{[bcd]} + h'_{[cda]} - h'_{[dab]} \quad \text{on} \quad U'_{[abcd]} \quad (5.9)$$

with $h'_{[abc]} = h'_{[abc]}(x, \lambda)$ holomorphic on $U'_{[abc]} \subset U'$. Notice that $h' = \hat{h}$. Moreover, we have another, smooth splitting $s'_{[abc]} := \sum_d f'_{[abcd]} \theta'_{[d]}$ obtained from the partition of unity. Evidently $h'_{[abc]} \neq s'_{[abc]}$, but both splittings are related by a gauge transformation $s'_{[abc]} = h'_{[abc]} - \varphi'_{[bc]} - \varphi'_{[ca]} - \varphi'_{[ab]}$ with

$$\varphi'_{[ab]} := \sum_c h'_{[abc]} \theta'_{[c]}. \quad (5.10)$$

We shall come back to this point below.

By the definition of a pull-back, $f' = \pi_1^* f$ is constant along the fibres of the fibration π_1 . That is, $d_{\pi_1} f' = 0 \Leftrightarrow V^A f'_{[abcd]} = 0$, where d_{π_1} is the relative exterior derivative introduced in (4.4). We now have the following natural differential forms on correspondence space:

$$\begin{aligned} A'_{[abc]} &:= d_{\pi_1} h'_{[abc]} \quad \text{on} \quad U'_{[abc]}, \\ B'_{[ab]} &:= \sum_c d_{\pi_1} h'_{[abc]} \wedge \bar{\partial} \theta'_{[c]} \quad \text{on} \quad U'_{[ab]}, \\ C'_{[a]} &:= \sum_{c,d} d_{\pi_1} h'_{[abc]} \wedge \bar{\partial} \theta'_{[b]} \wedge \bar{\partial} \theta'_{[c]} \quad \text{on} \quad U'_{[a]}. \end{aligned} \quad (5.11)$$

Notice that the differential forms $\pi_1^* \hat{A}_{[abc]}$, $\pi_1^* \hat{B}_{[ab]}$ and $\pi_1^* \hat{C}_{[a]}$ obtained from (5.7) are gauge equivalent to (5.11) via the gauge transformation that is mediated by the gauge parameter

(5.10). Furthermore as one may check, $A'_{[abc]}$ is Čech-closed, thus representing an element of $H^2(U', \Omega_{\pi_1}^1)$. However, by Proposition 4.2, this cohomology group vanishes and therefore, $A'_{[abc]}$ can be split holomorphically as $A'_{[abc]} = A'_{[ab]} + A'_{[bc]} + A'_{[ca]}$. As above, such a splitting is not unique, and this fact will turn out to correspond to space-time gauge transformations. The splitting of A' leads naturally to relative two-forms $\tilde{B}'_{[ab]} := d_{\pi_1} A'_{[ab]}$ which are gauge equivalent to $B'_{[ab]}$ in (5.11) by construction. We may now repeat this procedure once more as also $\tilde{B}'_{[ab]}$ is Čech-closed and $H^1(U', \Omega_{\pi_1}^2)$ vanishes again by virtue of Proposition 4.2. That is, there is a splitting $\tilde{B}'_{[ab]} = \tilde{B}'_{[a]} - \tilde{B}'_{[b]}$ yielding relative three-forms $\tilde{C}'_{[a]} := d_{\pi_1} \tilde{B}'_{[a]}$ which are gauge equivalent to $\tilde{C}'_{[a]}$ as given in (5.11). The relative three-forms $\tilde{C}'_{[a]}$ define a relative three-form \tilde{C}' via $\tilde{C}'_{[a]} = \tilde{C}'|_{U'_{[a]}}$, which is defined globally (i.e. on U'). Altogether, we have thus obtained a relative connective structure on the pull-back two-gerbe Γ' .

Note that the fibres of π_1 are three-dimensional and therefore, relative differential three-forms are automatically d_{π_1} -closed. This is a situation familiar from ambitwistor space without thickening. There, this fact implied that the spinor fields on space-time obtained by a push-forward from correspondence space do not satisfy any field equations [9]. The same is true here, as we shall see momentarily.

Using the Leray sequence and (4.16d) of Proposition 4.2, we find the identification

$$H^0(U', \Omega_{\pi_1}^3) \cong H^0(U, (\odot^2 S)[2]) . \quad (5.12)$$

Hence, any relative differential three-form \tilde{C}' can be expanded according to

$$\tilde{C}' = e_A \wedge e_B \wedge e_C \lambda_D \varepsilon^{ABCD} C^{EF} \lambda_E \lambda_F , \quad (5.13)$$

where $C^{AB} = C^{(AB)}$ is a spinor field living on space-time. In the above field expansion, we used relative one-forms e_A of homogeneity -1 , which combine with the tangent vectors V^A to the relative exterior derivative $d_{\pi_1} = e_A V^A$. The e_A are not unique, as the identity $\lambda_A V^A = 0$ implies that we can shift the e_A by terms proportional to λ_A . Notice that the above expansion of \tilde{C}' reflects this property.

The choice one has in the splittings of the Čech cocycles involved in the above construction result in \tilde{C}' 's that differ by d_{π_1} -exact relative three-forms, i.e. $\tilde{C}' \mapsto \tilde{C}' + d_{\pi_1} \tilde{\Lambda}'$, where $\tilde{\Lambda}' \in H^0(U', \Omega_{\pi_1}^2)$. Proposition 4.2 gives us $H^0(U', \Omega_{\pi_1}^2) \cong H^0(U, (S \otimes_{\mathcal{O}_U} S^\vee)_0[1])$ so that $\tilde{\Lambda}' = e_A \wedge e_B \lambda_C \varepsilon^{ABCD} \Lambda_D{}^E \lambda_E$, where $\Lambda_B{}^A$ is trace-less and depends only on space-time. A short calculation reveals that $\tilde{C}' \mapsto \tilde{C}' + d_{\pi_1} \tilde{\Lambda}'$ induces the space-time gauge transformation $C^{AB} \mapsto C^{AB} + \partial^{C(A} \Lambda^{B)}$. These considerations imply that C^{AB} should be regarded as an anti-self-dual three-form gauge potential. In fact, C^{AB} can be used to define a closed differential four-form via

$$G^A{}_B = \partial_{BC} C^{CA} , \quad (5.14a)$$

since such $G^A{}_B$ obey the Bianchi identity (which in spinorial notation reads as $\partial_{C[A}G^C{}_{B]} = 0 = \partial^{C[A}G^B{}_{C]}$). Explicitly, we may write $G_{MNKL} = \partial_{[M}C_{NKL]}$ with

$$G_{MNKL} := \sigma_{MNKL}{}^B{}_A G^A{}_B \quad \text{and} \quad C_{MNK} := \sigma_{MNKAB} C^{AB}. \quad (5.14b)$$

Here, the σ s are anti-symmetric products of the sigma-matrices defined in (A.8).

Altogether, we have shown that²⁷

$$H^3(\hat{U}, \mathcal{O}_{\hat{U}}) \cong \left\{ \frac{H^0(U, (\odot^2 S)[2])}{\partial^{AB} H^0(U, (S \otimes_{\mathcal{O}_U} S^\vee)_0[1])} \right\}. \quad (5.15)$$

Equivalently, holomorphic two-gerbes on the twistor space which are holomorphically trivial on any $\hat{x} \hookrightarrow \hat{U}$ are in one-to-one correspondence with holomorphically trivial two-gerbes on the correspondence space which come with a relative connective structure. Such two-gerbes are in turn in one-to-one correspondence with anti-self-dual three-form gauge potentials (modulo gauge transformations) on space-time. This is somewhat analogous to the Abelian Ward construction in four dimensions, where holomorphic line bundles on twistor space (i.e. zero-gerbes) subject to certain triviality conditions are in one-to-one correspondence with self-dual zero-rest-mass fields.

To make the construction of the differential forms (5.11) more explicit, let us discuss how one can find the holomorphic splitting (5.9) explicitly. For this, we shall follow the ideas of Helfer [47], who constructed similar holomorphic splittings in the context of self-dual Yang–Mills theory in four dimensions.

Recall that the smooth splitting $s' = \{s'_{[abc]}\}$ with $s'_{[abc]} = \sum_d f'_{[abcd]} \theta'_{[d]}$ and the holomorphic splitting $h' = \{h'_{[abc]}\}$ are related by a gauge transformation

$$s'_{[abc]} = h'_{[abc]} - \varphi'_{[abc]} \quad \text{with} \quad \varphi'_{[abc]} := \varphi'_{[ab]} + \varphi'_{[bc]} + \varphi'_{[ca]}, \quad (5.16)$$

cf. (5.10). Hence in order to find h' , we need to construct φ' , which is a solution to the differential equation

$$\bar{\partial} \varphi'_{[abc]} = - \sum_d f'_{[abcd]} \bar{\partial} \theta'_{[d]} =: a_{[abc]}^{(0,1)}. \quad (5.17)$$

Here, $\bar{\partial}$ is the Dolbeault operator on \mathbb{P}^3 and its Green function directly yields φ' .

Fortunately, this Green function has been computed before in [48] (see also [19, 18]). It is a $(0, 2)$ -form $G^{(0,2)} = G^{(0,2)}(\lambda_1, \lambda_2)$ on $\mathbb{P}^3 \times \mathbb{P}^3$ satisfying $\bar{\partial} G^{(0,2)} = \delta^{(0,3)}$, where the projective Dolbeault–Dirac delta distribution is given by

$$\delta^{(0,3)}(\lambda_1, \lambda_2) := \int_{\mathbb{C}} \frac{ds}{s} \delta^{(0,4)}(s\lambda_1 + \lambda_2). \quad (5.18)$$

²⁷under some mild topological assumptions, see below

Here, λ_i is short-hand notation for λ_{iA} and $\delta^{(0,4)}$ is the Dolbeault–Dirac delta distribution on \mathbb{C}^4 .²⁸ We have $\delta^{(0,3)}(t_1\lambda_1, t_2\lambda_2) = t_1^0 t_2^{-4} \delta^{(0,3)}(t_1\lambda_1, t_2\lambda_2)$ for $t_{1,2} \in \mathbb{C}^*$, and for holomorphic functions $f = f(\lambda)$ of homogeneity zero (defined on appropriate regions of \mathbb{P}^3) we have the identity

$$f(\lambda_1) = \int \Omega^{(3,0)}(\lambda_2) \wedge \delta^{(0,3)}(\lambda_1, \lambda_2) f(\lambda_2) , \quad (5.19)$$

where $\Omega^{(3,0)}(\lambda) := \frac{1}{4!} \varepsilon^{ABCD} \lambda_A d\lambda_B \wedge d\lambda_C \wedge d\lambda_D$ is the holomorphic measure on \mathbb{P}^3 . Notice that the Green function cannot be unique since one can always add a holomorphic $(0,2)$ -form to $G^{(0,2)}$ while preserving $\bar{\partial}G^{(0,2)} = \delta^{(0,3)}$. By virtue of Lemma 4.1, however, all holomorphic $(0,2)$ -forms (of any homogeneity) on $\mathbb{P}^3 \times \mathbb{P}^3$ must be Dolbeault-exact and therefore, $\bar{\partial}G^{(0,2)} = \delta^{(0,3)}$ enjoys a gauge freedom of the form $G^{(0,2)} \mapsto G^{(0,2)} + \bar{\partial}\varphi^{(0,1)}$ for some (smooth) differential $(0,1)$ -form $\varphi^{(0,1)}$ of appropriate homogeneity. A convenient choice of gauge is the Cachazo–Svrček–Witten gauge [49],

$$\bar{\xi}^A \frac{\partial}{\partial \bar{\lambda}_1^A} \lrcorner G^{(0,2)}(\lambda_1, \lambda_2) = 0 , \quad (5.20)$$

where bar denotes complex conjugation and ξ_A is some fixed reference spinor. In this gauge, the Green function is given by

$$G^{(0,2)}(\lambda_1, \lambda_2) = \int_{\mathbb{C}} \frac{ds}{s} \delta^{(0,3)}(s\xi + \lambda_1, \lambda_2) , \quad (5.21)$$

cf. e.g. [19]. It scales as $G^{(0,2)}(t_1\lambda_1, t_2\lambda_2) = t_1^0 t_2^{-4} G^{(0,2)}(t_1\lambda_1, t_2\lambda_2)$ for $t_{1,2} \in \mathbb{C}^*$, and it is invariant under re-scalings of the reference spinor ξ_A .

In order to compute the gauge parameter as a solution of (5.17), we choose a covering of \mathbb{P}^3 coming from U' and consider the respective triple overlaps. We shall assume that the reference spinor ξ_A is not contained in any of these triple overlaps.²⁹ Altogether, we arrive at the gauge parameter $\varphi'_{[abc]}$, which is given by the integral

$$\varphi'_{[abc]} = \int \Omega^{(3,0)}(\lambda_2) \wedge G^{(0,2)}(\lambda_1, \lambda_2) \wedge a^{(0,1)}_{[abc]}(\lambda_2) , \quad (5.22)$$

where we integrate over a suitable region.

Spinor fields from $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$, $h < 0$. As one may readily check, the above construction starting from elements of the cohomology group $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ for $h = -2$ can be straightforwardly extended to arbitrary $h < 0$. This is due to two facts. Firstly, Lemma 4.1 states that the restriction of the cohomology group $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ to \hat{x} is

²⁸On the complex plane \mathbb{C} described by the coordinate λ , the Dolbeault–Dirac distribution is given by $\delta^{(0,1)}(\lambda) = \bar{\partial}(2\pi i \lambda)^{-1}$. Therefore, $\delta^{(0,4)}(\lambda) = \delta^{(0,1)}(\lambda_{A=1}) \wedge \dots \wedge \delta^{(0,1)}(\lambda_{A=4})$.

²⁹This assumption is needed to avoid certain ‘error terms’, see e.g. the review [19] for details.

zero, and secondly, the cohomology groups $H^2(U', \Omega_{\pi_1}^1(-2h-4))$ and $H^1(U', \Omega_{\pi_1}^2(-2h-4))$ vanish by virtue of Proposition 4.2. Let us therefore directly state the final result:

Theorem 5.1. *As before, let $U \subset M^6$ be open and convex and set $U' := \pi_2^{-1}(U) \subset F^9$ and $\hat{U} := \pi_1(\pi_2^{-1}(U)) \subset P^6$, where $\pi_{1,2}$ are the maps in the double fibration (3.2). For $h \in -\frac{1}{2}\mathbb{N}$, there is a canonical isomorphism*

$$\mathcal{P} : H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)) \rightarrow \left\{ \frac{H^0(U, (\odot^{-2h-2}S)[2])}{\partial^{AB} H^0(U, (\odot^{-2h-3}S \otimes_{\mathcal{O}_U} S^\vee)_0[1])} \right\}. \quad (5.23)$$

Note that for $h = -\frac{1}{2}$ the right-hand-side is zero.

We would like to emphasise once more that for $h < 0$, the space-time interpretation of the twistor cohomology groups $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ leads to spinor fields that do not obey any field equations. This is in agreement with the results of Section 4.3. There we have seen that in order to obtain field equations in the case $h < 0$, one must thicken the twistor space appropriately into its ambient space \mathbb{P}_\circ^7 .

Spinor fields from $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-2))$, $h < 0$. Let us now come to the second type of cohomology group of interest, $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-2))$ for $h < 0$. As we explain momentarily, elements of this group will correspond to chiral zero-rest-mass fields on space-time.

To begin with, notice that the cohomology group $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-2))$ vanishes upon restriction to any \hat{x} when $h < 0$, by virtue of Lemma 4.1. Moreover, also the cohomology group $H^1(U', \Omega_{\pi_1}^1(-2h-2))$ vanishes as follows directly from Proposition 4.2. Therefore, we may repeat the steps we took above in the case of Čech three-cocycles. This yields a globally defined (i.e. on U') relative differential two-form \tilde{B}' of homogeneity $-2h-2$, which is d_{π_1} -closed. Contrary to the case of three-cocycles, however, the closedness of \tilde{B}' is not trivial, and it will imply certain differential equations on space-time. To work out the explicit form of these equations, we notice that

$$H^0(U', \Omega_{\pi_1}^2(-2h-2)) \cong H^0(U, (\odot^{-2h-1}S \otimes_{\mathcal{O}_U} S^\vee)_0[1]). \quad (5.24)$$

Using the relative one-forms e_A from before, we may express this explicitly as

$$\tilde{B}' = e_A \wedge e_B \lambda_C \varepsilon^{ABCD} B_D^{A_1 \dots A_{2|h|-1}} \lambda_{A_1} \dots \lambda_{A_{2|h|-1}}, \quad (5.25)$$

where $B_A^{A_1 \dots A_{2|h|-1}} = B_A^{(A_1 \dots A_{2|h|-1})}$ depends only on space-time and is totally trace-less. It is then a straightforward exercise to verify that $d_{\pi_1} \tilde{B}' = 0$ if and only if

$$\partial^{A(A_1} B_A^{A_2 \dots A_{2|h|})} = 0. \quad (5.26)$$

The induced gauge transformations are of the form $\tilde{B}' \mapsto \tilde{B}' + d_{\pi_1} \tilde{\Lambda}'$ with a gauge parameter $\tilde{\Lambda}' \in H^0(U', \Omega_{\pi_1}^1(-2h-2))$. Since

$$H^0(U', \Omega_{\pi_1}^1(-2h-2)) \cong H^0\left(U, \left(\frac{\odot^{-2h-1} S^\vee \otimes_{\mathcal{O}_U} S^\vee}{\odot^{-2h} S^\vee}\right)^\vee [1]\right) \quad (5.27)$$

by virtue of Proposition 4.2, we have

$$\tilde{\Lambda}' = e_A \lambda_B \varepsilon^{ABCD} \Lambda_{CD}^{A_1 \dots A_{2|h|-2}} \lambda_{A_1} \dots \lambda_{A_{2|h|-2}}, \quad (5.28)$$

where $\Lambda_{AB}^{A_1 \dots A_{2|h|-2}} = \Lambda_{[AB]}^{(A_1 \dots A_{2|h|-2})}$ depends only on space-time and is totally traceless. Therefore, $\tilde{B}' \mapsto \tilde{B}' + d_{\pi_1} \tilde{\Lambda}'$ corresponds on space-time to

$$B_B^{AA_1 \dots A_{2|h|-2}} \mapsto B_B^{AA_1 \dots A_{2|h|-2}} + \left[\partial_{CB} \Lambda^{C(AA_1 \dots A_{2|h|-2})} - \partial^{C(A} \Lambda_{CB}^{A_1 \dots A_{2|h|-2})} \right]_0, \quad (5.29)$$

where the subscript zero refers to the totally trace-less part. Note that the trace-part of $[\partial_{CB} \Lambda^{C(AA_1 \dots A_{2|h|-2})} - \partial^{C(A} \Lambda_{CB}^{A_1 \dots A_{2|h|-2})}]$ does not enter in (5.26) because the partial derivative is anti-symmetric in its indices.

From the potentials $B_A^{A_1 \dots A_{2|h|-1}}$, one can derive fields $H_{A_1 \dots A_{2|h|}} \in H^0(U, (\odot^{2|h|} S^\vee)[1])$ according to

$$H_{A_1 \dots A_{2|h|}} := \partial_{(A_1 B_1} \dots \partial_{A_{2|h|-1} B_{2|h|-1}} B_{A_{2|h|}}^{B_1 \dots B_{2|h|-1}}, \quad (5.30)$$

where the parentheses refer to the weighted symmetrisation of the indices $A_1, \dots, A_{2|h|}$. Because of (5.26), $H_{A_1 \dots A_{2|h|}}$ is a chiral zero-rest-mass field, i.e., it obeys the Dirac equation

$$\partial^{AA_1} H_{A_1 \dots A_{2|h|}} = 0. \quad (5.31)$$

As an example, consider the case $h = -1$. Here, B_A^B represents a two-form potential of a self-dual three-form field. Equation (5.26) then becomes $H^{AB} = \partial^{C(A} B_C^{B)} = 0$ and H_{AB} obeys the Dirac equation. Notice that in this case, the gauge transformation (5.29) reduces to the one for two-form potentials, $B_B^A \mapsto B_B^A + \partial^{AC} \Lambda_{CB} - \partial_{BC} \Lambda^{CA}$.

We may summarise our above discussions as follows.

Theorem 5.2. *Again, let $U \subset M^6$ be open and convex and set $U' := \pi_2^{-1}(U) \subset F^9$ and $\hat{U} := \pi_1(\pi_2^{-1}(U)) \subset P^6$, respectively. For $h \in -\frac{1}{2}\mathbb{N}$, there is a canonical isomorphism*

$$\mathscr{D} : H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-2)) \rightarrow H^0(U, \mathcal{Z}_s), \quad (5.32)$$

where $s = -h$.

Remark 5.1. Notice that for an element of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-2))$ with $h \geq 0$, one may write down the following contour integral

$$\psi_{A_1 \dots A_{2h}}(x) = \oint_{\mathcal{C}} \varepsilon^{ABCD} \lambda_A d\lambda_B \wedge d\lambda_C \frac{\partial}{\partial z^D} \hat{f}_{-2h-2}(z, \lambda) \Big|_{z=x \cdot \lambda} \lambda_{A_1} \cdots \lambda_{A_{2h}} . \quad (5.33)$$

Here, one has to make a choice of a contour $\mathcal{C} \hookrightarrow \mathbb{P}^3$, where \mathcal{C} is topologically a two-torus. The integral should be understood as a branched contour integral (see e.g. Penrose & Rindler [16] for a discussion in four dimensions). By differentiating under the integral, it is rather straightforward to see that $\psi_{A_1 \dots A_{2h}}$ obtained this way indeed obeys the appropriate zero-rest-mass field equation, i.e. the Dirac equation (and thus, the Klein–Gordon equation) for $h > 0$ and the Klein–Gordon equation for $h = 0$, respectively.

Remark 5.2. We have seen so far that elements³⁰ of both $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-6))$ and $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ correspond to $s = 1$ chiral zero-rest-mass fields. Let us now switch to the Dolbeault picture and represent elements of these cohomology groups in terms of their Dolbeault representatives. We shall denote them by $\hat{B}_0^{(0,2)}$ and $\hat{C}_{-6}^{(0,3)}$, respectively, where the subscript indicates the respective homogeneity. Next we extend these fields to off-shell fields and introduce the following twistor space action functional:

$$S = \int \Omega^{(6,0)} \wedge \hat{B}_0^{(0,2)} \wedge \bar{\partial} \hat{C}_{-6}^{(0,3)} . \quad (5.34)$$

Here, the holomorphic measure on twistor space is a $(6,0)$ -form of homogeneity $+6$,

$$\Omega^{(6,0)} := \oint_{\mathcal{C}} \frac{\Omega^{(4,0)}(z) \wedge \Omega^{(3,0)}(\lambda)}{z^A \lambda_A} , \quad (5.35)$$

where \mathcal{C} is any contour encircling $P^6 \hookrightarrow \mathbb{P}_o^7$. We have again used holomorphic volume forms $\Omega^{(4,0)}(z) := \frac{1}{4!} \varepsilon_{ABCD} dz^A \wedge dz^B \wedge dz^C \wedge dz^D$ and $\Omega^{(3,0)}(\lambda) := \frac{1}{4!} \varepsilon^{ABCD} \lambda_A d\lambda_B \wedge d\lambda_C \wedge d\lambda_D$.

The equations of motion resulting from the action (5.34) are $\bar{\partial} \hat{B}_0^{(0,2)} = 0 = \bar{\partial} \hat{C}_{-6}^{(0,3)}$. On-shell, $\hat{B}_0^{(0,2)}$ and $\hat{C}_{-6}^{(0,3)}$ therefore correspond indeed to representatives of the Čech cohomology groups $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ and $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-6))$.

³⁰Notice that $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ is not isomorphic to the elements of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}^*)$ with vanishing Dixmier–Douady class, since $H^2(\hat{U}, \mathbb{Z})$ is not zero. Hence, elements of $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ may not quite be regarded as one-gerbes. See also Remark 6.4.

It would be interesting to see if, after imposing appropriate reality conditions (and partially fixing gauge), the action (5.34) is related to one of the chiral space-time actions of Pasti, Sorokin & Tonin [33]. Finally, we note that it is rather straightforward to generalise the above action for any chiral field.

Remark 5.3. One might wonder what the Penrose–Ward transform would yield for holomorphic line bundles that become holomorphically trivial on any \hat{x} . Such line bundles are characterised by Čech one-cocycles $\hat{f} \in H^1(\hat{U}, \mathcal{O}_{\hat{U}})$ and therefore, we can use the above arguments to construct a global relative one-form \tilde{A}' that is d_{π_1} -closed. Since $H^0(U', \Omega_{\pi_1}^1) \cong H^0(U, S^\vee \wedge S^\vee)$, the expansion of \tilde{A}' is $\tilde{A}' = e_A \lambda_B \varepsilon^{ABCD} A_{CD}$, where A_{AB} depends only on space-time. Gauge transformations $\tilde{A}' \mapsto \tilde{A}' + d_{\pi_1} \Lambda$ for some holomorphic function Λ on U' (note that Λ depends only on space-time since global holomorphic functions on \mathbb{P}^3 are constant) reduce on space-time to $A_{AB} \mapsto A_{AB} + \partial_{AB} \Lambda$ and hence, A_{AB} is a Maxwell potential. The space-time equation that results from the flatness $d_{\pi_1} \tilde{A}' = 0$ is given by $\partial^{AC} A_{CB} - \partial_{BC} A^{CA} = 0$, as one may straightforwardly check. Thus, the Maxwell potential must be pure gauge. One may apply similar arguments for $H^1(\hat{U}, \mathcal{O}_{\hat{U}}(k))$. In summary, we may conclude that the first cohomology group appears not to give anything non-trivial.

6. Reduction to lower dimensions

In this section, we shall present various dimensional reductions of the twistor space P^6 to lower dimensions, specifically to twistor spaces of three- and four-dimensional space-times. Concretely, we shall focus on the reductions to the ambitwistor space P^5 [8, 9], to a new twistor space P^3 and to Hitchin’s minitwistor space P^2 [14]. We shall also discuss the corresponding Penrose and Penrose–Ward transforms. In particular, as is well-known, the ambitwistor space underlies a Penrose–Ward transform for the Maxwell equation in four dimensions, while the minitwistor space gives rise to a Penrose–Ward transform for the Abelian Bogomolny monopole equation in three dimensions.³¹ As we shall show below, the new twistor space P^3 underlies a Penrose–Ward transform for the Abelian self-dual string equation in four dimensions. We shall refer to P^3 as the *hyperplane twistor space*—the reason for this name becomes transparent shortly.

Starting from the double fibration (3.2), we shall quotient the twistor space P^6 , the correspondence space F^9 and space-time M^6 by various distributions to arrive at the

³¹Both ambitwistor space and minitwistor space can be used also in the non-Abelian setting, but in this paper, we are only interested in the Abelian setting.

following chain of double fibrations:

$$\begin{array}{ccccc}
 & & F^9 & & \\
 & \swarrow \pi_1 & \downarrow & \searrow \pi_2 & \\
 P^6 & & F^6 & & M^6 \\
 \downarrow & \swarrow \pi_3 & \downarrow \cong & \searrow \pi_4 & \downarrow \\
 P^5 & & F^6 & & M^4 \\
 \downarrow & \swarrow \pi_5 & \downarrow & \searrow \pi_6 & \downarrow \cong \\
 P^3 & & F^4 & & M^4 \\
 \downarrow & \swarrow \pi_7 & \downarrow & \searrow \pi_8 & \downarrow \\
 P^2 & & & & M^3
 \end{array} \tag{6.1}$$

where $M^n := \mathbb{C}^n$. To jump ahead of our story a bit, we shall find:

$$\begin{aligned}
 P^6 &\cong \Omega_{\mathbb{P}^3}^1 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \hat{=} \text{twistor space of six-dimensional space-time } M^6, \\
 P^5 &\cong \text{Jet}^1 \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \hat{=} \text{ambitwistor space of four-dimensional space-time } M^4, \\
 P^3 &\cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \hat{=} \text{hyperplane twistor space of four-dimensional space-time } M^4, \\
 P^2 &\cong \mathcal{O}_{\mathbb{P}^1}(2) \hat{=} \text{minitwistor space of three-dimensional space-time } M^3.
 \end{aligned}$$

The above terminology shall be clarified when constructing the respective twistor spaces.

6.1. Field equations in lower dimensions

Before presenting these reductions, we explain how the self-dual string equation, the Maxwell equation and the Bogomolny equation arise via dimensional reductions of the equations of motion of the six-dimensional self-dual three-form field strength $H = dB$. As we have already discussed in Section 2, a general three-form $H = dB$ in six dimensions is given by a pair of symmetric bi-spinors $H_{AB} = \partial_{C(A} B_{B)}^C$ and $H^{AB} = \partial^{C(A} B_{C}^{B)}$ via a (traceless) two-form potential B_B^A . Imposing self-duality onto H is equivalent to saying that $H^{AB} = 0$.

Field equations in four dimensions. To dimensionally reduce M^6 to M^4 , we split the $\mathbf{6} \cong \mathbf{4} \wedge \mathbf{4}$ representation of $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$ into the bi-fundamental representation $(\mathbf{2}, \mathbf{2})$ of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(4, \mathbb{C})$ plus twice the trivial representation to obtain

$$x^{AB} \rightarrow (x^{\alpha\dot{\beta}}, \varepsilon^{\alpha\beta} x^+, \varepsilon^{\dot{\alpha}\dot{\beta}} x^-), \tag{6.2}$$

where $\alpha, \beta, \dots, \dot{\alpha}, \dot{\beta}, \dots = 1, 2$. The symplectic forms $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$ of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ can be used to raise and lower $\mathfrak{sl}(2, \mathbb{C})$ spinor indices. Four-dimensional space-time M^4 is then given by the quotient $M^4 := M^6 / D_{M^6}$ with the distribution $D_{M^6} := \langle \partial_{\pm} \rangle$.

A two-form potential in six dimensions $B_B{}^A$ corresponding to a self-dual field strength reduces then to

$$B_B{}^A \rightarrow (A_{\alpha\dot{\alpha}}, B_{\alpha\beta} = B_{(\alpha\beta)}, B_{\dot{\alpha}\dot{\beta}} = B_{(\dot{\alpha}\dot{\beta})}, \phi) \quad (6.3)$$

and represents in four dimensions a one-form potential $A_{\alpha\dot{\alpha}}$, a two-form potential $(B_{\alpha\beta}, B_{\dot{\alpha}\dot{\beta}})$ and a scalar field ϕ . Notice that we used the symplectic forms $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ to rise and lower spinor indices. Correspondingly, gauge transformations of $B_B{}^A$,

$$B_B{}^A \mapsto B_B{}^A + \partial^{AC} \Lambda_{CB} - \partial_{BC} \Lambda^{CA} , \quad (6.4)$$

where $\Lambda_{AB} = -\Lambda_{BA}$, reduce in four dimensions to

$$\begin{aligned} A_{\alpha\dot{\alpha}} &\mapsto A_{\alpha\dot{\alpha}} + \partial_{\alpha\dot{\alpha}} \Lambda , \\ B_{\alpha\beta} &\mapsto B_{\alpha\beta} + \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_{(\alpha\dot{\alpha}} \Lambda_{\beta)\dot{\beta}} , \\ B_{\dot{\alpha}\dot{\beta}} &\mapsto B_{\dot{\alpha}\dot{\beta}} + \varepsilon^{\alpha\beta} \partial_{\alpha(\dot{\alpha}} \Lambda_{\beta)\dot{\beta}} , \\ \phi &\mapsto \phi . \end{aligned} \quad (6.5)$$

Furthermore, the (first-order) self-duality equation $H^{AB} = \partial^{C(A} B_C{}^{B)} = 0$ and the resulting (second-order) field equation $\partial^{AC} H_{CB} = 0$ reduce to

$$H_{\alpha\dot{\alpha}} := \varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} B_{\dot{\alpha}\dot{\gamma}} - \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} B_{\alpha\gamma} = \partial_{\alpha\dot{\alpha}} \phi \quad (6.6a)$$

and

$$\varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} f_{\dot{\alpha}\dot{\gamma}} + \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} f_{\alpha\gamma} = 0 , \quad (6.6b)$$

where $f_{\alpha\beta}$ and $f_{\dot{\alpha}\dot{\beta}}$ are the self-dual and anti-self-dual parts of the curvature of $A_{\alpha\dot{\alpha}}$,

$$f_{\alpha\beta} := \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_{(\alpha\dot{\alpha}} A_{\beta)\dot{\beta}} \quad \text{and} \quad f_{\dot{\alpha}\dot{\beta}} := \varepsilon^{\alpha\beta} \partial_{\alpha(\dot{\alpha}} A_{\beta)\dot{\beta}} . \quad (6.6c)$$

Equation (6.6a) is the self-dual string equation $H = \star_4 d\phi$ in spinor notation, while (6.6b) is the four-dimensional Maxwell equation. Notice that the Bianchi identity for the curvature of $A_{\alpha\dot{\alpha}}$ is given by

$$\varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} f_{\dot{\alpha}\dot{\gamma}} - \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} f_{\alpha\gamma} = 0 , \quad (6.7)$$

so that the equations for $f_{\alpha\beta}$ and $f_{\dot{\alpha}\dot{\beta}}$ decouple, i.e. $\varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} f_{\dot{\alpha}\dot{\gamma}} = 0 = \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} f_{\alpha\gamma}$.

Field equations in three dimensions. To further reduce to three dimensions, we split the $(\mathbf{2}, \mathbf{2})$ of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ into the $\mathbf{3} \oplus \mathbf{1}$ of $\mathfrak{sl}(2, \mathbb{C})$: $x^{\alpha\dot{\alpha}} \rightarrow (x^{\alpha\beta} = x^{(\alpha\beta)}, x^{[12]})$. We then have $M^3 := M^4/D_{M^4}$ with $D_{M^4} := \left\langle \frac{\partial}{\partial x^{[12]}} \right\rangle$. Here, the field strength of a gauge potential reduces directly according to

$$\partial_{\alpha\beta} A_{\gamma\delta} - \partial_{\gamma\delta} A_{\alpha\beta} = \varepsilon_{\alpha\gamma} f_{\beta\delta} + \varepsilon_{\beta\delta} f_{\alpha\gamma} , \quad (6.8)$$

and the BPS subsector of the reduced Maxwell equation is described by the Abelian Bogomolny equation³² $F = \star_3 d\phi$, which reads in spinor notation as

$$f_{\alpha\beta} = \partial_{\alpha\beta}\phi. \quad (6.9)$$

Note that this equation can be obtained from the self-dual string equation by defining $F = \frac{\partial}{\partial x^{[14]}} \lrcorner H$. In spinor notation, this amounts to defining $f_{\alpha\beta} := H_{(\alpha\beta)}$ and (6.6a) reduces to (6.9).

6.2. Ambitwistors and Maxwell fields

Ambitwistor space. The first reduction in the sequence (6.1) is that of (3.2) to the double fibration (π_3, π_4) containing the ambitwistor space P^5 . At the level of the twistor space P^6 , the splitting of $\mathfrak{sl}(4, \mathbb{C})$ into $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ which we used in reducing M^6 to M^4 in the previous section suggests that we decompose

$$(z^A, \lambda_A) \rightarrow (z^\alpha, -w^{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}). \quad (6.10)$$

Let us first consider the base space \mathbb{P}^3 of the fibration $P^6 \rightarrow \mathbb{P}^3$. To reduce \mathbb{P}^3 with homogeneous coordinates λ_A to $\mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates $(\mu_\alpha, \lambda_{\dot{\alpha}})$, we consider the corresponding reduction of the structure sheaf $\mathcal{O}_{\mathbb{P}^3}$ of \mathbb{P}^3 . Local sections f of $\mathcal{O}_{\mathbb{P}^3}$ fulfil the equation $\Upsilon f = 0$, where $\Upsilon := \lambda_A \frac{\partial}{\partial \lambda_A}$ is the Euler vector field on \mathbb{P}^3 . This reflects the invariance under re-scalings $\lambda_A \mapsto t\lambda_A$, with $t \in \mathbb{C}^*$. Local sections f of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ fulfil $\mu_\alpha \frac{\partial}{\partial \mu_\alpha} f = 0$ and $\lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} f = 0$. Therefore, the quotient of \mathbb{P}^3 by the distribution

$$D_{\mathbb{P}^3} := \left\langle \mu_\alpha \frac{\partial}{\partial \mu_\alpha} - \lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} \right\rangle \quad (6.11)$$

can be identified with $\mathbb{P}^1 \times \mathbb{P}^1$.

Analogously, one reduces \mathbb{P}^7 with homogeneous coordinates (z^A, λ_A) to $\mathbb{P}^3 \times \mathbb{P}^3$. Since we are interested in non-compact versions, let us directly remove the \mathbb{P}^3 defined by $z^A \neq 0$ and $\lambda_A = 0$ from \mathbb{P}^7 to obtain the ambient space $\mathbb{P}_\circ^7 = \mathbb{P}^7 \setminus \mathbb{P}^3 \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4$ of the twistor space P^6 we encountered before. The quotient of \mathbb{P}_\circ^7 by the distribution

$$D_{P^7} := \left\langle z^\alpha \frac{\partial}{\partial z^\alpha} + \lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} - w^{\dot{\alpha}} \frac{\partial}{\partial w^{\dot{\alpha}}} - \mu^\alpha \frac{\partial}{\partial \mu^\alpha} \right\rangle \quad (6.12)$$

can be identified with $\mathbb{P}_\circ^3 \times \tilde{\mathbb{P}}_\circ^3$, where \mathbb{P}_\circ^3 and $\tilde{\mathbb{P}}_\circ^3$ are each bi-holomorphic to the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2$. The quadric condition $z^A \lambda_A = 0$, which defines $P^6 \hookrightarrow \mathbb{P}_\circ^7$, descends to the quadric equation

$$z^\alpha \mu_\alpha - w^{\dot{\alpha}} \lambda_{\dot{\alpha}} = 0, \quad (6.13)$$

³²or Dirac monopole equation

which defines the ambitwistor space $P^5 \hookrightarrow \mathbb{P}_\circ^3 \times \tilde{\mathbb{P}}_\circ^3$ as a quadric hypersurface of $\mathbb{P}_\circ^3 \times \tilde{\mathbb{P}}_\circ^3$. Note that \mathbb{P}_\circ^3 is Penrose's twistor-space of four-dimensional space time while $\tilde{\mathbb{P}}_\circ^3$ is the dual twistor space.

The correspondence space F^6 is obtained as the quotient of $F^9 \cong \mathbb{C}^6 \times \mathbb{P}^3$ by the distribution

$$D_{F^9} := \left\langle \frac{\partial}{\partial x^\pm}, \mu_\alpha \frac{\partial}{\partial \mu_\alpha} - \lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} \right\rangle, \quad (6.14)$$

and we have $F^6 := F^9/D_{F^9} \cong \mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$. Altogether, we arrive at the following double fibration:

$$\begin{array}{ccc} & F^6 & \\ \pi_3 \swarrow & & \searrow \pi_4 \\ P^5 & & M^4 \end{array} \quad (6.15)$$

where π_4 is the trivial projection and

$$\pi_3 : (x^{\alpha\dot{\alpha}}, \lambda_{\dot{\alpha}}, \mu_\alpha) \mapsto (z^\alpha, w^{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}) = (x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}, x^{\alpha\dot{\alpha}}\mu_\alpha, \mu_\alpha, \lambda_{\dot{\alpha}}). \quad (6.16)$$

Note that the twistor distribution in this case is of rank one and generated by the vector field $\mu_\alpha \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}}$, i.e. $P^5 \cong F^6 / \langle \mu_\alpha \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \rangle$ with $\partial^{\alpha\dot{\alpha}} := \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial x^{\beta\dot{\beta}}}$.

Geometrically, a point x in four-dimensional space-time M^4 corresponds to a holomorphic embedding of $\hat{x} := \pi_3(\pi_4^{-1}(x)) \cong \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow P^5$. On the other hand, a point p in ambitwistor space P^5 corresponds to a null line $\pi_4(\pi_3^{-1}(x)) \hookrightarrow M^4$ given by

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \varrho \mu^\alpha \lambda^{\dot{\alpha}}, \quad \text{with } \varrho \in \mathbb{C}. \quad (6.17)$$

Moreover, if we introduce the two projections $\text{pr}_{1,2} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ to the first and second copy of \mathbb{P}^1 , respectively, and in addition

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l) &:= \text{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(l), \\ \Omega_{\mathbb{P}^1 \times \mathbb{P}^1}^p(k, l) &:= \Omega_{\mathbb{P}^1 \times \mathbb{P}^1}^p(\text{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(l)) \end{aligned} \quad (6.18)$$

for $k, l \in \mathbb{Z}$, then the sequence (3.9a) naturally reduces to a corresponding sequence for the ambitwistor space

$$0 \longrightarrow P^5 \longrightarrow (\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)) \otimes \mathbb{C}^2 \xrightarrow{\kappa} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \longrightarrow 0. \quad (6.19)$$

Here, $\kappa : (z^\alpha, w^{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}) \mapsto z^\alpha \mu_\alpha - w^{\dot{\alpha}} \lambda_{\dot{\alpha}}$. Upon dualising and twisting by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ the Euler sequence for $\mathbb{P}^1 \times \mathbb{P}^1$, we find

$$0 \longrightarrow \Omega_{\mathbb{P}^1 \times \mathbb{P}^1}^1(1, 1) \longrightarrow P^5 \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \longrightarrow 0. \quad (6.20)$$

This implies that P^5 can be identified with the bundle of first-order jets $\text{Jet}^1 \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ as a consequence of the jet-sequence

$$0 \longrightarrow \Omega_X^1(\mathcal{S}) \longrightarrow \text{Jet}^1 \mathcal{S} \longrightarrow \mathcal{S} \longrightarrow 0 \quad (6.21)$$

for an Abelian sheaf \mathcal{S} on a complex manifold X (see e.g. [12]).

Remark 6.1. *The above constructions show that we have a factorisation of the tangent bundle T_{M^4} into the two bundles of undotted and dotted chiral spinors. We shall denote these bundles by S and \tilde{S} and therefore, $T_{M^4} \cong S \otimes_{\mathcal{O}_{M^4}} \tilde{S}$, which is the reduction of the corresponding factorisation (2.1) in six dimensions. Note that such a factorisation amounts to choosing a holomorphic conformal structure. Furthermore, we shall make use of the following notation (for $k, l \in \mathbb{Z}$):*

$$[k, l] := \begin{cases} \otimes^k \det S^\vee \otimes_{\mathcal{O}_{M^4}} \otimes^l \det \tilde{S}^\vee & \text{for } k, l > 0, \\ \otimes^k \det S^\vee \otimes_{\mathcal{O}_{M^4}} \otimes^{|l|} \det \tilde{S} & \text{for } k > 0, l < 0, \\ \otimes^{|k|} \det S \otimes_{\mathcal{O}_{M^4}} \otimes^l \det \tilde{S}^\vee & \text{for } k < 0, l > 0, \\ \otimes^{|k|} \det S \otimes_{\mathcal{O}_{M^4}} \otimes^{|l|} \det \tilde{S} & \text{for } k, l < 0, \end{cases} \quad (6.22)$$

and we shall write $\mathcal{S}[k, l] := \mathcal{S} \otimes_{\mathcal{O}_{M^4}} [k, l]$ for an Abelian sheaf \mathcal{S} on M^4 . In addition, we introduce

$$\mathcal{O}_{P^5}(k, l) := \text{pr}^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l) \quad \text{for } k, l \in \mathbb{Z}, \quad (6.23)$$

where pr is the bundle projection $\text{pr} : P^5 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l)$ was defined in (6.18).

Penrose–Ward transform. Let us consider an open set $U \subset M^4$ and define $\hat{U} := \pi_3(\pi_4^{-1}(U)) \subset P^5$ with covering $\hat{\mathcal{U}} = \{\hat{U}_{[a]}\}$. We start from holomorphic line bundles over \hat{U} which are holomorphically trivial on any $\hat{x} \cong \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \hat{U}$. Such line bundles are characterised by Čech one-cocycles $\hat{f} = \{\hat{f}_{[ab]}\} \in H^1(\hat{U}, \mathcal{O}_{\hat{U}})$. The pull-back of \hat{f} to the correspondence space can be split holomorphically, $f'_{[ab]} = \pi_3^* \hat{f}_{[ab]} = h'_{[a]} - h'_{[b]}$. Since $f'_{[ab]}$ gets annihilated by the twistor distribution, we find $A' := \mu_\alpha \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} h'_{[a]}$ which is globally defined. Hence A' must be of the form $A' := \mu_\alpha \lambda_{\dot{\alpha}} A^{\alpha\dot{\alpha}}$, where $A_{\alpha\dot{\alpha}}$ depends only on space-time. Since the twistor distribution is one-dimensional, we do not obtain any space-time field equations for $A_{\alpha\dot{\alpha}}$. Moreover, since the splitting $f'_{[ab]} = h'_{[a]} - h'_{[b]}$ is not unique, we can always consider $h'_{[a]} \mapsto h'_{[a]} + \varphi'$, where φ' is defined globally on $\hat{U}' := \pi_4^{-1}(U) \subset F^6$. Therefore, φ' can only depend on space-time (since the \mathbb{P}^1 s are compact) and thus, it corresponds to transformations of the form $A_{\alpha\dot{\alpha}} \mapsto A_{\alpha\dot{\alpha}} + \partial_{\alpha\dot{\alpha}} \varphi'$.

In summary, this shows that $H^1(\hat{U}, \mathcal{O}_{\hat{U}})$ can be identified with the Maxwell potentials on U modulo gauge transformations. Notice that this construction also applies to the non-Abelian setting, that is, to Yang–Mills potentials.

In order to find a twistorial description of Maxwell/Yang–Mills fields which do satisfy the corresponding field equations, one has to do more work. Such descriptions (including their supersymmetric extensions) have been found a long time ago by Witten [8], Isenberg, Green & Yasskin [9, 10] and Manin [12]; see [41, 42] for a cohomological analysis, Mason & Skinner [50] for an action principle for Yang–Mills theory on ambitwistor space, and [51] for a recent review in conventions similar to ours. We therefore keep the following discussion brief.

As is well-known, in order to construct self-dual (or anti-self-dual) solutions to the Maxwell/Yang–Mills equations, one employs Ward’s construction [5] starting from holomorphic vector bundles over Penrose’s twistor space \mathbb{P}^3_{\circ} (or the dual twistor space $\tilde{\mathbb{P}}^3_{\circ}$) subject to certain triviality conditions. Because ambitwistor space incorporates both twistors and dual twistors, it can be used to give a twistor interpretation of the Maxwell/Yang–Mills equations. As we have seen above, however, the ambitwistor space itself is not quite sufficient to recover these equations. To resolve this problem, one needs to thicken the ambitwistor space into its ambient space $\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}$ to a certain order. This is fully analogous to the thickening of P^6 in \mathbb{P}^7_{\circ} as encountered in Section 4.3. The ℓ -th order thickening (or the ℓ -th infinitesimal neighbourhood) is defined by $P^5_{[\ell]} := (P^5, \mathcal{O}_{\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}} / \mathcal{I}^{\ell+1})$. Here, $\mathcal{O}_{\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}}$ is the sheaf of holomorphic functions on $\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}$ and \mathcal{I} is the ideal subsheaf of $\mathcal{O}_{\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}}$ consisting of those functions that vanish on P^5 . Now we have the following theorem:

Theorem 6.1. ([8, 9]) *Let U be an open subset of M^4 such that any null line intersects U in a convex set. Then there is a one-to-one correspondence between gauge equivalence classes of complex holomorphic solutions to the Yang–Mills equations on U and equivalence classes of holomorphic vector bundles which are holomorphically trivial on any $\hat{x} \cong \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow P^5$ for all $x \in U$ and which admit an extension to a 3-rd order thickening $P^5_{[3]}$ of P^5 in $\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}$.*

Note that if the holomorphic vector bundle can be extended to a finite neighbourhood within the ambient space $\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}$, then the space-time gauge field constructed from this vector bundle is either self-dual, anti-self-dual or Abelian [9]. Thus, if one is only interested in the Maxwell equation (as we are in the present case) one may work with holomorphic line bundles on the ambient space $\mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}$ which are holomorphically trivial on $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3_{\circ} \times \tilde{\mathbb{P}}^3_{\circ}$. Since \mathbb{P}^3_{\circ} is Penrose’s twistor space while $\tilde{\mathbb{P}}^3_{\circ}$ its dual, one finds a self-dual and an anti-self dual field strength. Both can be linearly superposed to obtain a solution to the Maxwell equation. This is possible as the equations for the two helicities decouple, as

discussed in Section 6.1.

Penrose transform. Besides the Penrose–Ward transform and Maxwell fields, one may also consider other spinor fields on space-time which can be obtained from certain cohomology groups on ambitwistor space. In fact, we have the following theorems due to Pool [42] and Eastwood [52].

Theorem 6.2. (Pool [42]) *Consider the double fibration (6.15). Let $U \subset M^4$ be open and convex and set $U' := \pi_4^{-1}(U) \subset F^6$ and $\hat{U} := \pi_3(\pi_4^{-1}(U)) \subset P^3$. For $h_{1,2} \in \frac{1}{2}\mathbb{N}$, there is a canonical isomorphism*

$$\begin{aligned} \mathcal{P} : H^1(\hat{U}, \mathcal{O}_{\hat{U}}(2h_1 - 2, 2h_2 - 2)) &\rightarrow \\ &\rightarrow \left\{ \frac{H^0(U, (\odot^{2h_1-1} S \otimes_{\mathcal{O}_U} \odot^{2h_2-1} \tilde{S})[1, 1])}{\partial^{\alpha\dot{\alpha}} H^0(U, \odot^{2h_1-2} S \otimes_{\mathcal{O}_U} \odot^{2h_2-2} \tilde{S})} \right\}. \end{aligned} \quad (6.24)$$

In particular, for $h_1 = h_2 = 1$ we recover the identification of $H^1(\hat{U}, \mathcal{O}_{\hat{U}})$ with Maxwell potentials on $U \subset M^4$ modulo gauge transformations. Pool’s result in four dimensions is very similar to Theorem 5.1 in six dimensions.

Theorem 6.3. (Eastwood [52]) *Let $U \subset M^4$ be open and convex and set $U' := \pi_4^{-1}(U) \subset F^6$ and $\hat{U} := \pi_3(\pi_4^{-1}(U)) \subset P^3$, where the maps $\pi_{3,4}$ are those appearing in the double fibration (6.15). For $h_{1,2} \in \frac{1}{2}\mathbb{N}_0$, there is a canonical isomorphism*

$$\begin{aligned} \mathcal{P} : H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h_1 - 2, -2h_2 - 2)) &\rightarrow \\ &\rightarrow \left\{ \begin{array}{l} \psi_{\alpha_1 \dots \alpha_{2h_1} \dot{\alpha}_1 \dots \dot{\alpha}_{2h_2}} \in H^0(U, (\odot^{2h_1} S \otimes_{\mathcal{O}_U} \odot^{2h_2} \tilde{S})[1, 1]) \\ \text{such that } \partial^{\alpha_1 \dot{\alpha}_1} \psi_{\alpha_1 \dots \alpha_{2h_1} \dot{\alpha}_1 \dots \dot{\alpha}_{2h_2}} = 0 \end{array} \right\}. \end{aligned} \quad (6.25)$$

Eastwood’s result is particularly interesting for the case when $h_1 = h_2 = \frac{1}{2}$ since then, we have an identification of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-3, -3))$ with all conserved currents on $U \subset M^4$. Notice that Eastwood also gives a twistorial interpretation of *massive* space-time fields in [52].

6.3. Hyperplane twistors and self-dual strings

In this section, we introduce a new twistor space P^3 which, to our knowledge, has not been considered before. For that reason, we shall present a more detailed discussion in the following. As we shall see below, this twistor space underlies a Penrose–Ward transform mapping a certain cohomology group on P^3 to solutions to the self-dual string equation on M^4 in a bijective manner.

Hyperplane twistor space. Here, both space-time and correspondence space are the same as those in the double fibration (6.15) containing the ambitwistor space P^5 . The *hyperplane twistor space* P^3 is obtained by quotienting P^5 by the distribution

$$D_{P^5} := \left\langle \mu^\alpha \frac{\partial}{\partial z^\alpha}, \lambda^{\dot{\alpha}} \frac{\partial}{\partial w^{\dot{\alpha}}} \right\rangle. \quad (6.26)$$

It is rather straightforward to see that $P^3 := P^5/D_{P^5}$ is bi-holomorphic to the total space of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Altogether, we may write down the following double fibration:

$$\begin{array}{ccc} & F^6 & \\ \pi_5 \swarrow & & \searrow \pi_6 \\ P^3 & & M^4 \end{array} \quad (6.27)$$

where π_6 is the trivial projection and

$$\pi_5 : (x^{\alpha\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}) \mapsto (z, \mu_\alpha, \lambda_{\dot{\alpha}}) = (x^{\alpha\dot{\alpha}} \mu_\alpha \lambda_{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}). \quad (6.28)$$

Note that the twistor distribution is of rank three and generated by the vector fields $\mu_\alpha \partial^{\alpha\dot{\alpha}}$ and $\lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}}$, i.e. $P^3 \cong F^6 / \langle \mu_\alpha \partial^{\alpha\dot{\alpha}}, \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \rangle$, with $\partial^{\alpha\dot{\alpha}} := \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial x^{\beta\dot{\beta}}}$ as before.

The geometric twistor correspondence here is as follows. By virtue of the incidence relation $z = x^{\alpha\dot{\alpha}} \mu_\alpha \lambda_{\dot{\alpha}}$, a point $x \in M^4$ corresponds to a holomorphic embedding of $\hat{x} := \pi_5(\pi_6^{-1}(x)) \cong \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow P^3$, while a point $p \in P^3$ corresponds to a hyperplane $\pi_6(\pi_5^{-1}(p)) \hookrightarrow M^4$ in space-time. To see this, note that the incidence relation $z = x^{\alpha\dot{\alpha}} \mu_\alpha \lambda_{\dot{\alpha}}$ can be solved for fixed $p = (z, \mu, \lambda) \in P^3$ by

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \mu^\alpha \nu^{\dot{\alpha}} + \kappa^\alpha \lambda^{\dot{\alpha}}. \quad (6.29)$$

Here, $x_0^{\alpha\dot{\alpha}}$ is a particular solution and κ_α and $\nu_{\dot{\alpha}}$ are some unconstrained spinors, which parametrise translations of $x_0^{\alpha\dot{\alpha}}$ along totally null two-planes (the α - and β -planes). These two kinds of planes intersect along the null-line $x_0^{\alpha\dot{\alpha}} + \mu^\alpha \lambda^{\dot{\alpha}}$, and thus, the solutions (6.29) describe indeed a three-dimensional hyperplane in M^4 . The apparent four parameters in the spinors $\nu^{\dot{\alpha}}$ and κ^α are reduced to three, because the shifts

$$\kappa_\alpha \mapsto \kappa_\alpha + \varrho \mu_\alpha \quad \text{and} \quad \nu_{\dot{\alpha}} \mapsto \nu_{\dot{\alpha}} - \varrho \lambda_{\dot{\alpha}} \quad \text{for} \quad \varrho \in \mathbb{C} \quad (6.30)$$

leave the solution (6.29) invariant.

Remark 6.2. The above constructions show again that we have a factorisation of the tangent bundle T_{M^4} into the two bundles of undotted and dotted chiral spinors. As before, we shall denote these bundles by S and \tilde{S} and therefore, $T_{M^4} \cong S \otimes_{\mathcal{O}_{M^4}} \tilde{S}$. Similarly to Remark 6.1, we introduce

$$\mathcal{O}_{P^3}(k, l) := \text{pr}^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l) \quad \text{for} \quad k, l \in \mathbb{Z}, \quad (6.31)$$

where pr is the bundle projection $\text{pr} : P^3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l)$ was defined in (6.18).

Penrose–Ward transform. Let us fix a Stein cover $\hat{\mathcal{U}} = \{\hat{U}_{[a]}\}$ of P^3 together with a partition of unity $\hat{\theta} = \{\hat{\theta}_{[a]}\}$ subordinate to $\hat{\mathcal{U}}$. As before, we shall use the abbreviations $\hat{U}_{[ab]} = \hat{U}_{[a]} \cap \hat{U}_{[b]}$, etc. for intersections of patches. Instead of working with all of P^3 , we again allow for the restriction to open neighbourhoods \hat{U} corresponding to open sets $U \subset M^4$ via $\hat{U} = \pi_5^{-1}(U)$.

Consider a representative \hat{f} of the cohomology group $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$. Using the partition of unity, we have a smooth splitting of \hat{f} ,

$$\hat{f}_{[abc]} = \hat{s}_{[ab]} + \hat{s}_{[bc]} + \hat{s}_{[ca]} \quad (6.32)$$

on $\hat{U}_{[abc]}$, where $\hat{s} = \{\hat{s}_{[ab]} := \sum_c \hat{f}_{[abc]} \hat{\theta}_{[c]}\}$. As usual, we perform the transition to the Dolbeault picture by introducing certain differential $(0, 1)$ - and $(0, 2)$ -forms on $\hat{U}_{[ab]}$ and $\hat{U}_{[a]}$, respectively:

$$\hat{A}_{[ab]} := \bar{\partial} \hat{s}_{[ab]} = \sum_c \hat{f}_{[abc]} \bar{\partial} \hat{\theta}_{[c]} \quad \text{and} \quad \hat{B}_{[a]} := \sum_{b,c} \hat{f}_{[abc]} \bar{\partial} \hat{\theta}_{[b]} \wedge \bar{\partial} \hat{\theta}_{[c]}. \quad (6.33)$$

Notice that the two-form $\hat{B}^{(0,2)} := \{\hat{B}_{[a]}\}$ is globally defined (i.e. on all of \hat{U}), since $\hat{B}_{[a]} = \sum_{bc} \hat{s}_{[bc]} \bar{\partial} \hat{\theta}_{[b]} \wedge \bar{\partial} \hat{\theta}_{[c]}$. Notice also that the corresponding curvature $(0, 3)$ -form $\hat{H}^{(0,3)} := \bar{\partial} \hat{B}^{(0,2)}$ is zero. Altogether, we have obtained a holomorphic connective structure. Since $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ vanishes when restricted to any $\hat{x} \cong \mathbb{P}^1 \times \mathbb{P}^1$ for any $x \in U$, we can find a holomorphic splitting of \hat{f} on $\hat{U}_{[abc]} \cap \hat{x}$.

The next step in the Penrose–Ward transform is to pull-back \hat{f} to the correspondence space F^6 , which yields $f' = \pi_5^* \hat{f}$. Analogously to the six-dimensional setting, we shall here make use of the relative p -forms along $\pi_5 : F^6 \rightarrow P^3$. We denote them by $\Omega_{\pi_5}^p$ and they are given by a short exact sequence

$$0 \longrightarrow \pi_5^* \Omega_{P^3}^1 \wedge \Omega_{F^6}^{p-1} \longrightarrow \Omega_{F^6}^p \longrightarrow \Omega_{\pi_5}^p \longrightarrow 0. \quad (6.34)$$

This sequence also yields the projection $\text{pr}_{\pi_5} : \Omega^p \rightarrow \Omega_{\pi_5}^p$ via the quotient mapping, which allows us to introduce the relative differential $d_{\pi_5} := \text{pr}_{\pi_5} \circ d : \Omega_{\pi_5}^p \rightarrow \Omega_{\pi_5}^{p+1}$, cf. (4.4). On $U' = \pi_5^{-1}(\hat{U}) = \pi_6^{-1}(U)$, we choose the cover $\mathcal{U}' := \{U'_{[a]}\}$ that is induced by the cover $\hat{\mathcal{U}}$.

Next we observe that the Čech cocycle $f' = \pi_5^* \hat{f}$ can be split holomorphically on $U'_{[abc]}$. Therefore, we have

$$f'_{[abc]} = h'_{[ab]} + h'_{[bc]} + h'_{[ca]}, \quad (6.35)$$

where $h' = \{h'_{[ab]}\}$ is holomorphic and, as before, its choice is not unique. Since f' is the pull-back of \hat{f} via π_5 , it is annihilated by the relative exterior derivative d_{π_5} . This allows us to introduce the following differential forms:

$$A'_{[ab]} := d_{\pi_5} h'_{[ab]} \quad \text{on } U'_{[ab]} \quad \text{and} \quad B'_{[a]} := \sum_b d_{\pi_5} h'_{[ab]} \wedge \bar{\partial} \theta'_{[b]} \quad \text{on } U'_{[a]}. \quad (6.36)$$

Similarly to the six-dimensional construction, one can check that the relative differential one-form $A'_{[ab]}$ is Čech-closed and thus represents an element of $H^1(U', \Omega^1_{\pi_5})$. To compute this cohomology group, we point out that we have a similar representation of the relative p -forms in terms of certain pull-back sheaves as in the six-dimensional setting, cf. Proposition 4.1. If we let $\Omega^p_{\pi_5}(k, l) := \Omega^p_{\pi_5} \otimes_{\mathcal{O}_{F6}} \pi_5^* \mathcal{O}_{P^3}(k, l)$, then one can check that $\Omega^p_{\pi_5}$ is characterised by the following short exact sequence:

$$\begin{aligned} 0 \longrightarrow \Omega^p_{\pi_5} \longrightarrow \Lambda^p [\pi_6^*(S[1, 1]) \otimes_{\mathcal{O}_{F6}} \pi_5^* \mathcal{O}_{P^3}(0, 1) \oplus \\ \oplus \pi_6^*(\tilde{S}[1, 1]) \otimes_{\mathcal{O}_{F6}} \pi_5^* \mathcal{O}_{P^3}(1, 0)] \longrightarrow \\ \longrightarrow \pi_6^*[1, 1] \otimes_{\mathcal{O}_{F6}} \Omega^{p-1}_{\pi_5}(1, 1) \longrightarrow 0, \end{aligned} \quad (6.37)$$

where we have used the notation introduced in Remarks 6.1 and 6.2. The induced long exact sequence of cohomology groups yields that $H^1(U', \Omega^1_{\pi_5}) = 0$ since the map

$$\begin{aligned} H^0(U', \pi_6^*(S[1, 0]) \otimes_{\mathcal{O}_{U'}} \pi_5^* \mathcal{O}_{P^3}(0, 1) \oplus \pi_6^*(\tilde{S}[0, 1]) \otimes_{\mathcal{O}_{U'}} \pi_5^* \mathcal{O}_{P^3}(1, 0)) \\ \rightarrow H^0(U', \pi_6^*[1, 1] \otimes_{\mathcal{O}_{U'}} \Omega^0_{\pi_5}(1, 1)) \end{aligned} \quad (6.38)$$

is surjective. Therefore, we have a splitting $A'_{[ab]} = A'_{[a]} - A'_{[b]}$, which allows us to define a relative two-form by setting $\tilde{B}'_{[a]} := d_{\pi_5} A'_{[a]}$. By construction, the two-forms $B'_{[a]}$ and $\tilde{B}'_{[a]}$ are gauge equivalent. Moreover, $\tilde{B}'_{[a]}$ defines a global (i.e. on U') relative two-form \tilde{B}' , which is relatively closed, i.e. $d_{\pi_5} \tilde{B}' = 0$. Altogether, we have obtained a flat relative connective structure on the correspondence space. The equation $d_{\pi_5} \tilde{B}' = 0$ is not vacuous since the fibres of π_5 are three-dimensional and hence, relative differential two-forms are not automatically d_{π_5} -closed. As we shall see momentarily, this equation is equivalent to the self-dual string equation on space-time.

The final step in the construction is to push-down $\tilde{B}' \in H^0(U', \Omega^2_{\pi_5})$ to space-time. To this end, we first point out that the sequence (6.37) also yields the isomorphism

$$H^0(U', \Omega^2_{\pi_5}) \cong H^0(U, (\odot^2 S \otimes_{\mathcal{O}_U} \Lambda^2 \tilde{S} \oplus \Lambda^2 S \otimes_{\mathcal{O}_U} \odot^2 \tilde{S} \oplus \Lambda^2 S \otimes_{\mathcal{O}_U} \Lambda^2 \tilde{S})[2, 2]), \quad (6.39)$$

which implies that a relative two-form corresponds to the space-time fields $B_{\alpha\beta} = B_{(\alpha\beta)}$, $B_{\dot{\alpha}\dot{\beta}} = B_{(\dot{\alpha}\dot{\beta})}$ and ϕ representing the field content of the self-dual string. To show that these fields indeed obey the self-dual string equation (6.6a), let us work in local coordinates

$(x^{\alpha\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}})$ on the correspondence space. Recall that the relative tangent bundle of the fibration $\pi_5 : F^6 \rightarrow P^3$ (respectively, the twistor distribution) is spanned by the vector fields $V^\alpha := \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}}$ and $V^{\dot{\alpha}} := \mu_\alpha \partial^{\alpha\dot{\alpha}}$, which satisfy $\mu_\alpha V^\alpha = \lambda_{\dot{\alpha}} V^{\dot{\alpha}}$. Correspondingly, the relative cotangent bundle, i.e. the bundle of relative one-forms, will be spanned by one-forms e_α and $e_{\dot{\alpha}}$ subject to the equivalence relation

$$(e_\alpha, e_{\dot{\alpha}}) \sim (e_\alpha + \mu_\alpha e, e_{\dot{\alpha}} - \lambda_{\dot{\alpha}} e) \quad (6.40)$$

for some e to accommodate $\mu_\alpha V^\alpha = \lambda_{\dot{\alpha}} V^{\dot{\alpha}}$. The relative exterior derivative is then given by $d_{\pi_5} = e_\alpha V^\alpha + e_{\dot{\alpha}} V^{\dot{\alpha}}$. In terms of the e_α and $e_{\dot{\alpha}}$, the most general expansion of a relative two-form then reads as

$$\begin{aligned} \tilde{B}' = & e^\alpha \wedge e_\alpha \lambda^{\dot{\alpha}} \lambda^{\dot{\beta}} B_{\dot{\alpha}\dot{\beta}} + e^{\dot{\alpha}} \wedge e_{\dot{\alpha}} \mu^\alpha \mu^\beta B_{\alpha\beta} + \\ & + 2e^\alpha \wedge e^{\dot{\alpha}} \left(\mu_\alpha \lambda^{\dot{\beta}} B_{\dot{\alpha}\dot{\beta}} - \lambda_{\dot{\alpha}} \mu^\beta B_{\alpha\beta} - \mu_\alpha \lambda_{\dot{\alpha}} \phi \right), \end{aligned} \quad (6.41)$$

where all the fields $B_{\alpha\beta}$, $B_{\dot{\alpha}\dot{\beta}}$ and ϕ depend only on $x^{\alpha\dot{\alpha}}$ and represent the isomorphism (6.39). Note that the expression (6.41) is invariant under (6.40) as required for consistency. Using this expansion together with the expression for the relative exterior derivative, we obtain after some algebra that $d_{\pi_5} \tilde{B}' = 0$ is equivalent to

$$\varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} B_{\dot{\alpha}\dot{\gamma}} - \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} B_{\alpha\gamma} = \partial_{\alpha\dot{\alpha}} \phi \quad (6.42)$$

on space-time. This, however, is precisely the self-dual string equation (6.6a). Finally, we would like to point out that the above splittings are as always not unique, which results in gauge transformations in \tilde{B}' of the form $\tilde{B}' \mapsto \tilde{B}' + d_{\pi_5} \tilde{\Lambda}'$ where $\tilde{\Lambda}' \in H^0(U', \Omega_{\pi_5}^1)$. Such $\tilde{\Lambda}'$ are of the form $\tilde{\Lambda}' = (e_\alpha \lambda_{\dot{\alpha}} + e_{\dot{\alpha}} \mu_\alpha) \Lambda^{\alpha\dot{\alpha}}$, where $\Lambda^{\alpha\dot{\alpha}} = \Lambda^{\alpha\dot{\alpha}}(x)$ depends only on space-time. Then $\tilde{B}' \mapsto \tilde{B}' + d_{\pi_5} \tilde{\Lambda}'$ induces the space-time gauge transformations displayed in (6.5).

In summary, we have proved the following theorem:

Theorem 6.4. *Consider the double fibration (6.27). Let $U \subset M^4$ be open and convex and set $U' := \pi_6^{-1}(U) \subset F^6$ and $\hat{U} := \pi_5(\pi_6^{-1}(U)) \subset P^3$, respectively. Then there is a canonical isomorphism:*

$$H^2(\hat{U}, \mathcal{O}_{\hat{U}}) \cong \left\{ \begin{array}{l} \text{gauge equivalence classes of complex holomorphic solutions} \\ \text{to the self-dual string equation on } U \end{array} \right\}. \quad (6.43)$$

Note that similarly to the discussion presented in Section 4, also here one may construct the holomorphic splitting (6.35) explicitly by using the Green function of the Dolbeault operator on $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 6.3. One may also consider the case of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h_1-2, -2h_2-2))$ for $h_{1,2} \leq -1$ using the above constructions. Specifically, one would obtain $B_{\dot{\alpha}_1 \dots \dot{\alpha}_{-2h_2-2} \dot{\beta}_1 \dots \dot{\beta}_{-2h_1}}, B_{\alpha_1 \dots \alpha_{-2h_1-2} \dot{\beta}_1 \dots \dot{\beta}_{-2h_2}} and $\phi_{\alpha_1 \dots \alpha_{-2h_1-2} \dot{\beta}_1 \dots \dot{\beta}_{-2h_2-2}}$ as space-time fields, which are totally symmetric in all of their spinor indices. Likewise, the self-dual string equation generalises straightforwardly to$

$$\begin{aligned} \varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{(\alpha_1 \dot{\beta}} B_{\alpha_2 \dots \alpha_{-2h_1-1}) \dot{\gamma} \dot{\beta}_1 \dots \dot{\beta}_{-2h_2-1}} - \varepsilon^{\beta\gamma} \partial_{\beta(\dot{\beta}_1} B_{\dot{\beta}_2 \dots \dot{\beta}_{-2h_2-1}) \gamma \alpha_1 \dots \alpha_{-2h_1-1}} \\ = \partial_{(\alpha_1(\dot{\beta}_1 \phi_{\alpha_2 \dots \alpha_{-2h_1-1}) \dot{\beta}_2 \dots \dot{\beta}_{-2h_2-1})} . \end{aligned} \quad (6.44)$$

The cases with either $h_1 = -\frac{1}{2}$ or $h_2 = -\frac{1}{2}$ seem not to give anything non-trivial. The cases with $h_{1,2} \geq 0$ will be discussed below.

Remark 6.4. One might be tempted to identify the Čech cocycles \hat{f} on twistor space with holomorphic one-gerbes that become holomorphically trivial on \hat{x} . This is, however, not quite accurate. Recall that holomorphic one-gerbes are characterised by elements of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}^*)$ and that the exponential sheaf sequence (5.1) implies

$$H^2(\hat{U}, \mathbb{Z}) \longrightarrow H^2(\hat{U}, \mathcal{O}_{\hat{U}}) \longrightarrow H^2(\hat{U}, \mathcal{O}_{\hat{U}}^*) \xrightarrow{\text{DD}} H^3(\hat{U}, \mathbb{Z}) . \quad (6.45)$$

Now the group $H^3(\hat{U}, \mathbb{Z})$ vanishes, which implies that all one-gerbes on the hyperplane twistor space have vanishing Dixmier–Douady class and therefore become holomorphically trivial on any \hat{x} . However, $H^2(\hat{U}, \mathbb{Z})$ does not vanish and thus, $H^2(\hat{U}, \mathcal{O}_{\hat{U}}^*)$ cannot be identified with $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$.

Remark 6.5. One might wonder what the Penrose–Ward transform would yield for holomorphic line bundles over P^3 which are holomorphically trivial on any \hat{x} . Such line bundles are characterised by Čech one-cocycles $\hat{f} = \{\hat{f}_{[ab]}\} \in H^1(\hat{U}, \mathcal{O}_{\hat{U}})$. The pull-back of \hat{f} can be split holomorphically, $f'_{[ab]} = \pi_5^* \hat{f}_{[ab]} = h'_{[a]} - h'_{[b]}$. This allows us to introduce a (global) relative one-form $A' = e_\alpha A^\alpha + e_{\dot{\alpha}} A^{\dot{\alpha}}$ with components $A^\alpha := V^\alpha h'_{[a]} =: \lambda_{\dot{\alpha}} A^{\alpha\dot{\alpha}}$ and $A^{\dot{\alpha}} := V^{\dot{\alpha}} h'_{[a]} =: \mu_\alpha A^{\alpha\dot{\alpha}}$, as a consequence of $\lambda_{\dot{\alpha}} A^{\dot{\alpha}} = \mu_\alpha A^\alpha$. Here, $A_{\alpha\dot{\alpha}}$ depends only on space-time. From the flatness condition on the corresponding curvature, we obtain the equation $V_\alpha A_{\dot{\alpha}} - V_{\dot{\alpha}} A_\alpha = \lambda^{\dot{\alpha}} \mu^\beta (\partial_{\alpha\dot{\alpha}} A_{\beta\dot{\alpha}} - \partial_{\beta\dot{\alpha}} A_{\alpha\dot{\alpha}}) = 0$. Hence, $A_{\alpha\dot{\alpha}}$ has to be pure gauge. Note that this even holds true in the non-Abelian case for rank- r holomorphic vector bundle over P^3 with $r > 1$.

Penrose transform. So far, we have discussed the Penrose–Ward transform yielding the identification of $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ with the moduli space of solutions (obtained from the solution space as a quotient with respect to the group of gauge transformations) of the self-dual string equation. In this paragraph we would like to demonstrate that there is a natural

extension to other field equations in four dimensions. Specifically, we are interested in space-time fields of the form $\psi_{\alpha_1 \dots \alpha_{2h_1} \dot{\alpha}_1 \dots \dot{\alpha}_{2h_2}} = \psi_{(\alpha_1 \dots \alpha_{2h_1})(\dot{\alpha}_1 \dots \dot{\alpha}_{2h_2})}$ with $h_{1,2} \in \frac{1}{2}\mathbb{N}_0$ which obey

$$\partial^{\alpha_1 \dot{\beta}} \psi_{\alpha_1 \dots \alpha_{2h_1} \dot{\alpha}_1 \dots \dot{\alpha}_{2h_2}} = 0 = \partial^{\beta \dot{\alpha}_1} \psi_{\alpha_1 \dots \alpha_{2h_1} \dot{\alpha}_1 \dots \dot{\alpha}_{2h_2}} . \quad (6.46a)$$

We shall refer to such fields as zero-rest-mass fields of helicity (h_1, h_2) . When either h_1 or h_2 vanishes then we have chiral spinors, $\psi_{\alpha_1 \dots \alpha_{2h_1}}$, or anti-chiral spinors, $\psi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2h_2}}$. In the special case $h_1 = h_2 = 0$, we have a scalar field (denoted by ϕ) and, as always, a second-order field equation

$$\square \phi = 0 . \quad (6.46b)$$

Such zero-rest-mass fields can be constructed from representatives of cohomology groups on twistor space P^3 via the following theorem:

Theorem 6.5. *Consider the double fibration (6.27). Let $U \subset M^4$ be open and convex and set $U' := \pi_6^{-1}(U) \subset F^6$ and $\hat{U} := \pi_5(\pi_6^{-1}(U)) \subset P^3$. For $h_{1,2} \in \frac{1}{2}\mathbb{N}_0$, there is a canonical isomorphism³³*

$$\mathcal{P} : H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h_1 - 2, -2h_2 - 2)) \rightarrow \left\{ \begin{array}{l} \text{zero-rest-mass fields} \\ \text{of helicity } (h_1, h_2) \text{ on } U \end{array} \right\} . \quad (6.47)$$

The proof of this theorem is essentially the same as the one presented in Section 4.2. One can use again the machinery of spectral sequences and the direct image sheaves computed from the short exact sequences (6.37) of the relative differential forms after twisting by the pull-back of $\mathcal{O}_{P^3}(k, l)$ for appropriate k and l . Therefore, we shall refrain from repeating analogous arguments for the present situation, but concern ourselves with the corresponding contour integral formulæ, which will make the theorem more transparent.

To this end, let us consider the canonical four-patch covering of P^3 . An element of the cohomology group $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h_1 - 2, -2h_2 - 2))$ can be represented by a collection of four functions $\hat{f} = \{\hat{f}_{[abc]}\}$, each of which is of homogeneity $(-2h_1 - 2, -2h_2 - 2)$. For our choice of cover, all four triple overlaps $U_{[abc]}$ are of the same topology and moreover, also two double overlaps have the same topology as the triple overlaps. Therefore, two of the four functions defining the cocycle \hat{f} can be fixed using equivalence relations, i.e. adding an appropriate Čech one-cochain. One of the remaining two functions is then fixed by the cocycle condition for f , leaving us with just one function defining the cocycle \hat{f} .

³³Notice that for either $h_1 = 0$ or $h_2 = 0$ one has a reduction of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h_1 - 2, -2h_2 - 2))$ to $H^1(\hat{U}_1, \mathcal{O}_{\hat{U}_1}(-2h_1 - 2))$ or $H^1(\hat{U}, \mathcal{O}_{\hat{U}}(-2h_2 - 2))$, where \hat{U}_1 is an open subset of Penrose's twistor space \mathbb{P}^3_0 and \hat{U}_2 of its dual $\tilde{\mathbb{P}}^3_0$.

Let us denote this function by $\hat{f}_{-2h_1-2,-2h_2-2} = \hat{f}_{-2h_1-2,-2h_2-2}(z, \mu, \lambda)$. Using now the holomorphic measure on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\Omega^{(2,0)} := \frac{1}{4} \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \mu_\alpha d\mu_\beta \wedge \lambda_{\dot{\alpha}} d\lambda_{\dot{\beta}} , \quad (6.48)$$

we can define spinor fields via the following integral:

$$\psi_{\alpha_1 \dots \alpha_{2h_1} \dot{\alpha}_1 \dots \dot{\alpha}_{2h_2}}(x) = \oint_{\mathcal{C}} \Omega^{(2,0)} \mu_{\alpha_1} \dots \mu_{\alpha_{2h_1}} \lambda_{\dot{\alpha}_1} \dots \lambda_{\dot{\alpha}_{2h_2}} \hat{f}_{-2h_1-2,-2h_2-2}(x^{\beta\dot{\beta}} \mu_\beta \lambda_{\dot{\beta}}, \mu, \lambda) . \quad (6.49)$$

It is trivial to check that these spinor fields obey the field equations (6.46). Notice that one can write this covariantly using branched contour integrals (see e.g. Penrose & Rindler [16] for a discussion in the ordinary twistor setting). Notice also that the above integral is very similar to Eastwood's integral in the ambitwistor setting [52].

6.4. Minitwistors and monopoles

Minitwistor space. The twistor space used to describe monopoles on three-dimensional space-time $M^3 := \mathbb{C}^3$ is Hitchin's minitwistor space P^2 [14]. It can be regarded as the tangent space of \mathbb{P}^1 or, equivalently, the total space of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1}(2) \rightarrow \mathbb{P}^1$.

In the twistor picture, the restriction of the moduli space of sections from M^4 to M^3 amounts to restricting the line bundle P^3 to the diagonal \mathbb{P}^1 with $\mu_\alpha = \lambda_\alpha$ in the base $\mathbb{P}^1 \times \mathbb{P}^1$ of P^3 . We can achieve this by quotienting by the distribution

$$D_{P^3} := \left\langle \mu^\beta \lambda_\beta \left(\lambda_\alpha \frac{\partial}{\partial \mu_\alpha} - \mu_\alpha \frac{\partial}{\partial \lambda_\alpha} \right) \right\rangle . \quad (6.50)$$

That is, $P^2 := P^3/D_{P^3}$, and the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ reduces to the line bundle $\mathcal{O}_{\mathbb{P}^1}(2) \rightarrow \mathbb{P}^1$. The correspondence space is obtained by taking the quotient of F^6 by the distribution

$$D_{F^6} = \left\langle \frac{\partial}{\partial x^{[12]}}, \mu^\beta \lambda_\beta \left(\lambda_\alpha \frac{\partial}{\partial \mu_\alpha} - \mu_\alpha \frac{\partial}{\partial \lambda_\alpha} \right) \right\rangle , \quad (6.51)$$

so that $F^4 := F^6/D_{F^6} \cong \mathbb{C}^3 \times \mathbb{P}^1$. Here, we have the double fibration

$$\begin{array}{ccc} & F^4 & \\ \pi_7 \swarrow & & \searrow \pi_8 \\ P^2 & & M^3 \end{array} \quad (6.52)$$

with $\pi_7 : (x^{\alpha\beta}, \mu_\alpha) \mapsto (z, \mu_\alpha) = (x^{\alpha\beta} \mu_\alpha \mu_\beta, \mu_\alpha)$ and π_8 being the trivial projection. In the case of P^2 , we have a geometric twistor correspondence between points in M^3 and

holomorphic embeddings $\mathbb{P}^1 \hookrightarrow P^2$, as well as between points in P^2 and (oriented) lines in M^3 . Notice that the twistor distribution here is of rank two and it is generated by the vector fields $\mu_\alpha \partial^{\alpha\beta}$, i.e. $P^2 \cong F^4 / \langle \mu_\alpha \partial^{\alpha\beta} \rangle$ with $\partial^{\alpha\beta} := \varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta} \frac{\partial}{\partial x^{\gamma\delta}}$.

Remark 6.6. *There is an alternative way of obtaining the minitwistor space from the ambitwistor space in the non-Abelian setting. Firstly, one reduces to the miniambitwistor space [53] underlying a Penrose–Ward transform for solutions to the three-dimensional Yang–Mills–Higgs theory. Restricting to BPS solutions then amounts to restricting the miniambitwistor space to the minitwistor space.*

Penrose–Ward transform. The construction of the Abelian monopole equations in the twistor context has been discussed extensively in the literature e.g. in [14, 54]; see also [55] and [56] for a review in conventions similar to ours. Let us therefore just make a few comments in the following.

The Penrose–Ward transform works here in the familiar way. A holomorphic vector bundle over P^2 which becomes holomorphically trivial upon restriction to the submanifolds $\hat{x} \cong \mathbb{P}^1 \hookrightarrow P^2$ can be pulled back to F^4 . Specifically, we have $\hat{f} = \{\hat{f}_{[ab]}\} \in H^1(\hat{U}, \mathcal{O}_{\hat{U}})$ for $\hat{U} \subset P^2$. The pull-back of \hat{f} can be split holomorphically, $f'_{[ab]} = \pi_7^* \hat{f}_{[ab]} = h'_{[a]} - h'_{[b]}$. Using the Liouville theorem, this allows us to introduce a global relative one-form A' with components

$$A'^\alpha := \mu_\beta \partial^{\alpha\beta} h'_{[a]} =: \mu_\beta (A^{\alpha\beta} - \varepsilon^{\alpha\beta} \phi), \quad (6.53)$$

where the fields on the right-hand-side depend only on space-time. From the flatness condition on the corresponding curvature, we obtain

$$f_{\alpha\beta} = \partial_{\alpha\beta} \phi, \quad (6.54)$$

where $f_{\alpha\beta}$ is the curvature of $A_{\alpha\beta}$ as given in (6.8). This is the spinorial form of the Bogomolny monopole equation $F := dA = \star_3 d\phi$ in three dimensions.

7. Conclusions and outlook

We have discussed both Penrose and Penrose–Ward transforms on the twistor space P^6 associated with flat six-dimensional space-time. We have proved that these transforms yield bijections between certain cohomological data on P^6 and solutions to the chiral zero-rest-mass field equations on space-time. We have also reviewed various contour integral formulæ yielding chiral zero-rest-mass-fields. We have pointed out that some of them implicitly require an infinitesimal extension of the twistor space P^6 into its ambient space

$\mathbb{P}_\circ^7 = \mathbb{P}^7 \setminus \mathbb{P}^3 \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4$. Interestingly, it is also possible to establish a Penrose–Ward transform for both Čech three-cocycles and two-cocycles with values in various sheaves on twistor space. The Čech three-cocycles yield free potentials modulo gauge transformations, as one might have expected from analogous results on ambitwistor space and the contour integrals. The Čech two-cocycles, however, lead to potentials giving rise to zero-rest-mass fields on space-time, which satisfy the appropriate field equations. Lifting their corresponding Dolbeault representatives to off-shell fields on twistor space, we were able to give a twistor space action principle for them. This action is to be seen analogous to holomorphic Chern–Simons theory in the twistor description of self-dual fields on four-dimensional space-time.

We found that dimensional reductions of chiral spinor-fields from six-dimensional space-time to four and three dimensions yield corresponding reductions of twistor space P^6 and the cohomological data in question. In particular, we have shown that the quadric P^6 in the open subset \mathbb{P}_\circ^7 of \mathbb{P}^7 reduces to the ambitwistor space P^5 , which is a quadric in the open subset $\mathbb{P}_\circ^3 \times \tilde{\mathbb{P}}_\circ^3$ of $\mathbb{P}^3 \times \tilde{\mathbb{P}}^3$. Moreover, the infinitesimal extension of P^6 to \mathbb{P}_\circ^7 that we encountered in the integral formulæ is mapped to the familiar infinitesimal extension of the ambitwistor space P^5 inside $\mathbb{P}_\circ^3 \times \tilde{\mathbb{P}}_\circ^3$. Together with our observations in the context of the contour integral formulæ, this suggests that a Penrose–Ward transform yielding solutions to the field equations should be constructed on a thickened or even supersymmetrically extended version of the twistor space that we have considered here. We have also found the novel hyperplane twistor space $P^3 \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. The Penrose–Ward transform for this case maps very naturally Čech two-cocycles with values in the structure sheaf over the hyperplane twistor space to solutions to the self-dual string equation. This result is particularly interesting, as the self-dual string has been very rarely discussed in the context of integrable field theories. Moreover, it fills a gap by providing, at least in the Abelian case, the twistor picture underlying the generalised Atiyah–Drinfeld–Hitchin–Manin–Nahm construction of [25]. We have been able to complement this by a Penrose transform as well as corresponding contour integral formulæ.

Our results lead to a number of important questions that are beyond the scope of this paper, but which we shall address in forthcoming publications [57]. The most obvious issue to be resolved is that of thickenings of the twistor space P^6 . As indicated in Section 4.3, it could be preferable to work directly with a supertwistor space containing P^6 as its body, instead of thickenings of P^6 . This is suggested by our experience with the corresponding four-dimensional discussions. A supersymmetric extension of the twistor space P^6 which

has some of the desired properties has been proposed by Chern [30].³⁴ On the field theory side, it remains to perform an analysis of the supersymmetric field equations in terms of constraint equations of a certain superconnection along the lines of [58, 59]. Such an analysis has recently been worked out for the M2-brane models by Samtleben & Wimmer [60, 61].

A full analysis of the twistor space action (5.34) that appeared very naturally from the Dolbeault representatives of the cohomology groups of interest in the Penrose–Ward transform is also necessary. It would be exciting if it was possible to establish a connection to the actions describing self-dual three-forms in six dimensions as given by Pasti, Sorokin & Tonin [33]. As indicated, this would require imposing reality conditions, thickening and partially fixing gauges.

Moreover, it is certainly of major importance to find non-Abelian extensions of the Penrose–Ward transforms we presented here, possibly based on the non-Abelian generalisations of gerbes as introduced e.g. in [62] or [63]. The fact that a non-Abelian version of the self-dual string equation can be found on loop space [25] indeed suggests that non-Abelian generalisations should exist. Amongst other things, an advantage of this approach is that, given a non-Abelian Čech cohomology on twistor space, all space-time structures like gauge transformations, field equations, etc. follow automatically.

Another interesting issue to address concerns the hidden (non-local) symmetries of chiral six-dimensional field theories using twistor constructions as the ones presented here. Specifically, one would be interested in the deformation theory of the corresponding cohomological data on twistor space to obtain hidden symmetries and integrable hierarchies for the equations of motion on space-time. Similar constructions were performed in four dimensions e.g. in [64]. Here, one would be mostly interested in the non-Abelian case.

All of the above issues, i.e. the supersymmetric and non-Abelian extensions as well as the action principle and hidden symmetries and integrable hierarchies, should also be studied in the case of the hyperplane twistor space. In this case, we expect some of these problems to have more straightforward solutions than in the case of P^6 .

One might also want to develop a twistor description of the $\mathcal{N} = (1, 1)$ supersymmetric Yang–Mills theory in six dimensions. In fact, Samtleben & Wimmer [61] (see Devchand [65] for an earlier account) gave a Lax formulation for the constraint equations of the superconnection of this theory (see Harnad & Shnider [59]) with a six-dimensional null-vector as spectral parameter. This implies that one should be considering the space of all (supersymmetric) null-rays in six-dimensional (supersymmetric) space-time for a twistorial formulation of this theory [66].

³⁴The dimensional reduction to superambitwistor space, however, suggests to use a slightly different supertwistor space.

Acknowledgements. We are very grateful to M. Dunajski, J. Figueroa-O’Farrill, L. Hughston, L. Mason, J. McOrist, S. Palmer, A. Torrielli, and R. Wimmer for helpful discussions and comments. C.S. was supported by an EPSRC Career Acceleration Fellowship. M.W. was in part supported by an STFC Postdoctoral Fellowship at DAMTP, University of Cambridge and by a Senior Research Fellowship at Wolfson College Cambridge.

Appendix

A. Spinor notation and reality conditions in six and four dimensions

Six dimensions. We wish to describe a six-dimensional chiral field theory containing a self-dual three-form field strength. Similarly to four dimensions, where self-duality of a two-form field strength is only possible in Euclidean and split signature (Kleinian signature), we also have a limited choice of signatures in six dimensions: The Hodge star on three-forms is only idempotent for Minkowski and split signature. That is, we have to restrict ourselves to the spaces $\mathbb{R}^{p,6-p} := (\mathbb{R}^6, \eta)$ with p odd and metric $\eta = (\eta_{MN}) = \text{diag}(-\mathbb{1}_p, \mathbb{1}_{6-p})$ for $M, N, \dots = 1, \dots, 6$.

Consider now the generators γ_M of the Clifford algebra $\mathcal{Cl}(\mathbb{R}^{p,6-p})$ of $\mathbb{R}^{p,6-p}$ with p odd. We thus have $\{\gamma_M, \gamma_N\} = -2\eta_{MN}$. The spinor representation $\mathbf{8}_s$ of the corresponding generators γ_{MN} of $\text{Spin}(p, 6-p)$ splits into the direct sum $S \oplus \tilde{S}$ of the subspaces of chiral and anti-chiral spinors. There is a natural isomorphism between \tilde{S} and S^\vee [16], so that we can exclusively work with elements of S and S^\vee . We label the corresponding spinors by upper and lower indices, e.g. $(\psi^A) \in S$ and $(\chi_A) \in S^\vee$ with $A, B, \dots = 1, \dots, 4$.

We can choose a representation of $\mathcal{Cl}(\mathbb{R}^{p,6-p})$ such that the sigma-matrices³⁵ σ_{AB}^M and $\tilde{\sigma}^{MAB}$ are antisymmetric and satisfy $\tilde{\sigma}^{MAB} = \frac{1}{2}\varepsilon^{ABCD}\sigma_{CD}^M$. By definition, we have

$$\sigma_{AC}^M \tilde{\sigma}^{NCB} + \sigma_{AC}^N \tilde{\sigma}^{MCB} = -2\eta^{MN}\mathbb{1}_4. \quad (\text{A.1})$$

We use the sigma-matrices to convert back and forth between vector and spinor notation:

$$x^{AB} := \tilde{\sigma}_M^{AB} x^M \quad \text{and} \quad x^M = \frac{1}{4}\sigma_{AB}^M x^{AB}, \quad (\text{A.2})$$

so that we can use the Levi-Civita symbol ε_{ABCD} as a metric:

$$x^M x_M = \frac{1}{4}x_{AB}x^{AB}, \quad \text{with} \quad x_{AB} = x_M \sigma_{AB}^M = \frac{1}{2}\varepsilon_{ABCD}x^{CD}. \quad (\text{A.3})$$

Derivatives act according to

$$\frac{\partial}{\partial x^{AB}} x^{CD} = \frac{1}{4}\sigma_{AB}^M \frac{\partial}{\partial x^M} x^{CD} = \frac{1}{2}(\delta_A^C \delta_B^D - \delta_A^D \delta_B^C). \quad (\text{A.4})$$

³⁵They are the generalisations of the sigma-matrices in four dimensions, i.e. they correspond to the non-vanishing off-diagonal blocks in the Clifford generators.

For signature $(p, 6-p) = (1, 5)$, a set of sigma-matrices is given by

$$x_{AB} = x_M \sigma_{AB}^M = \begin{pmatrix} 0 & x_0 + x_5 & -x_3 - ix_4 & -x_1 + ix_2 \\ -x_0 - x_5 & 0 & -x_1 - ix_2 & x_3 - ix_4 \\ x_3 + ix_4 & x_1 + ix_2 & 0 & -x_0 + x_5 \\ x_1 - ix_2 & -x_3 + ix_4 & x_0 - x_5 & 0 \end{pmatrix}, \quad (\text{A.5a})$$

while for signature $(p, 6-p) = (3, 3)$ we may use

$$x_{AB} = x_M \sigma_{AB}^M = \begin{pmatrix} 0 & x_0 + x_5 & -x_3 - ix_4 & -x_2 - ix_1 \\ -x_0 - x_5 & 0 & x_2 - ix_1 & x_3 - ix_4 \\ x_3 + ix_4 & -x_2 + ix_1 & 0 & -x_0 + x_5 \\ x_2 + ix_1 & -x_3 + ix_4 & x_0 - x_5 & 0 \end{pmatrix}. \quad (\text{A.5b})$$

On spinors $\psi \in S$ and $\chi \in S^\vee$, we can introduce an antilinear map

$$\tau_p(\psi, \chi) = (C_p \psi^*, C_p^{-1} \chi^*), \quad \text{with} \quad C_p := \begin{pmatrix} 0 & 1 & 0 & 0 \\ p-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & p-2 & 0 \end{pmatrix}. \quad (\text{A.6})$$

Here, ψ^* and χ^* denote the complex conjugate of the spinors ψ and χ ; $p-2 = \pm 1$ depending on the two cases $p = 1$ and $p = 3$. Notice that with our choice of sigma-matrices, $\tau_p(\sigma^{MN} \lambda) = (\sigma^{MN})^* \tau_p(\lambda)$ and therefore τ_p yields a real structure on spinors. This induces the following reality conditions on the bi-spinors $(x^{AB}) \in T_{M^6} \cong S \wedge S$:

$$x^{AB} = (C_p)^A{}_C x^{CD} (C_p^{-1})_D{}^B. \quad (\text{A.7})$$

We define weighted, totally antisymmetric products of sigma-matrices according to their index structure, for example:

$$\begin{aligned} \sigma^{MNA}{}_B &:= \frac{1}{2} (\sigma^{MAC} \sigma_{CB}^N - \sigma^{NAC} \sigma_{CB}^M) = -\sigma^{MN}{}_B{}^A, \\ \sigma^{MNKAB} &:= \frac{1}{3!} (\sigma^{MAC} \sigma_{CD}^N \sigma^{KDB} \pm \text{permutations}) = \sigma^{MNKBA}, \\ \sigma_{AB}^{MNK} &:= \frac{1}{3!} (\sigma_{AC}^M \sigma^{NCD} \sigma_{DB}^K \pm \text{permutations}) = \sigma_{BA}^{MNK}, \\ \sigma^{MNKLA}{}_B &:= \frac{1}{4!} (\sigma^{MAC} \sigma_{CD}^N \sigma^{KDE} \sigma_{EB}^L \pm \text{permutations}) = \sigma^{MNKLA}{}_B. \end{aligned} \quad (\text{A.8})$$

These products are used in the translation between vector and spinor notation for differ-

ential forms:³⁶

$$\begin{aligned}
A_M &\rightarrow A_{AB} = \frac{1}{4}\sigma_{AB}^M A_M, \\
B_{MN} &\rightarrow B_B^A = \sigma^{MN}{}_B^A B_{MN}, \\
H_{MNK} &\rightarrow (H_{AB}, H^{AB}) = (\sigma_{AB}^{MNK} H_{MNK}, \sigma^{MNK AB} H_{MNK}), \\
G_{MNKL} &\rightarrow G_B^A = \sigma^{MNKL A}{}_B G_{MNKL}.
\end{aligned} \tag{A.9}$$

Notice that the products σ_{AB}^{MNK} and σ_{MNK}^{AB} form projectors onto self-dual and anti-self-dual three-forms, respectively:

$$\sigma^{MNK AB} = -\frac{1}{3!}\varepsilon^{MNK RST}\sigma_{RST}^{AB} \quad \text{and} \quad \sigma_{MNK}^{AB} = -\frac{1}{3!}\varepsilon_{MNK RST}\sigma^{RST AB}, \tag{A.10}$$

where $\varepsilon^{012345} = +1$ and $\varepsilon_{012345} := -1$.

Four dimensions. The six-dimensional spaces $\mathbb{R}^{p,p-6}$ allow for dimensional reductions to four dimensions with arbitrary signature. The process is always the same. For brevity, we focus here on the reduction $\mathbb{R}^{p,q}$ to $\mathbb{R}^{p-1,q-1}$, which is done by imposing $\frac{\partial}{\partial x^0} = \frac{\partial}{\partial x^5} = 0$. In spinor notation, this amounts to a real form of the branching of the isometry group corresponding to $\text{SL}(4, \mathbb{C}) \rightarrow \text{SL}(2, \mathbb{C}) \times \widetilde{\text{SL}(2, \mathbb{C})}$. Correspondingly, we restrict the sigma-matrices to their off-diagonal blocks:

$$\sigma_{AB}^M \rightarrow \begin{pmatrix} 0 & \sigma_{\alpha\dot{\alpha}}^\mu \\ -\sigma_{\alpha\dot{\alpha}}^\mu & 0 \end{pmatrix}, \tag{A.11}$$

where $\mu, \nu, \dots = 1, \dots, 4$ and $\alpha, \beta, \dots, \dot{\alpha}, \dot{\beta}, \dots = 1, 2$. The translation between vector and spinor notation then reads as

$$x^{\alpha\dot{\alpha}} := \sigma_\mu^{\alpha\dot{\alpha}} x^\mu \iff x^\mu = \frac{1}{2}\sigma_{\alpha\dot{\alpha}}^\mu x^{\alpha\dot{\alpha}}. \tag{A.12}$$

Recall that indices can be raised and lowered with the antisymmetric tensor of $\mathfrak{sl}(2, \mathbb{C})$: $\sigma^{\mu\alpha\dot{\alpha}} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma_{\beta\dot{\beta}}^\mu$. We use the conventions $\varepsilon_{12} = 1$ and $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma$. The norm-squared of x^μ is thus given by

$$x^\mu x_\mu = \frac{1}{2}x^{\alpha\dot{\alpha}}x_{\alpha\dot{\alpha}} := \frac{1}{2}x^{\alpha\dot{\alpha}}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}x^{\beta\dot{\beta}}. \tag{A.13}$$

The real structures we obtain from reducing the six-dimensional real structure read as

$$\tau_p(\psi, \chi) = (C_p \psi^*, C_p^{-1} \chi^*) \quad \text{with} \quad C_p := \begin{pmatrix} 0 & 1 \\ p-2 & 0 \end{pmatrix}, \tag{A.14}$$

³⁶In spinor notation, two- and four-forms actually correspond to pairs $(B^A{}_B, B_A{}^B)$ and $(G^A{}_B, G_A{}^B)$. Depending on the choice of sigma-matrices, i.e. the choice of the underlying space-time, there are relations within these pairs, which allow for the distinction between two- and four-forms on space-time. To simplify our formulæ, we use the above identification of spinor fields.

and for the coordinate vector, we have

$$x^{\alpha\dot{\alpha}} = (C_p)^\alpha{}_\beta x^{\beta\dot{\beta}} (C_p^{-1})_{\dot{\beta}}{}^{\dot{\alpha}} \Rightarrow x^{1\dot{1}} = \bar{x}^{2\dot{2}} \quad \text{and} \quad x^{1\dot{2}} = (p-2)\bar{x}^{2\dot{1}}. \quad (\text{A.15})$$

B. Some remarks about n -gerbes

Recall that the transition functions of a smooth principal $\text{U}(1)$ -bundle³⁷ over a manifold X form a Čech one-cocycle with values in the sheaf of invertible smooth functions on X denoted by \mathcal{E}_X^* . Thus, a principal $\text{U}(1)$ -bundle is a geometric realisation of an element in $H^1(X, \mathcal{E}_X^*)$. For a holomorphic $\text{U}(1)$ -bundle, the relevant sheaf is that of invertible holomorphic functions on X , \mathcal{O}_X^* , and the transition functions form an element of $H^1(X, \mathcal{O}_X^*)$.

The n -gerbes defined in [67], or rather the *bundle n -gerbes* developed in [68], form geometric realisations of the cohomology groups $H^{n+1}(X, \mathcal{E}_X^*)$ in the smooth category and $H^{n+1}(X, \mathcal{O}_X^*)$ in the holomorphic category. Note that in the smooth category, we always have the isomorphism³⁸

$$H^{n+1}(X, \mathcal{E}_X^*) \cong H^{n+2}(X, \mathbb{Z}), \quad (\text{B.1})$$

which identifies the Čech $(n+2)$ -cocycle defining the n -gerbe with its characteristic class.

Explicitly, the geometric realisation of a Čech $(n+2)$ -cocycle is done by introducing an $(n-1)$ -gerbe over the fibre product $Y \times_X Y$ of some surjective submersion $Y \rightarrow X$. The space Y could be the disjoint union of the patches $\{U_{[a]}\}$ of a cover of X , for example. We are not interested in these constructions themselves, and merely prefer the term n -gerbe over the term Čech $(n+1)$ -cocycle just as we prefer the term line bundle over the term transition functions.

Let us now consider the examples $n = 0, 1, 2$ relevant in our discussion. In the following, we assume that we have a cover $\mathfrak{U} = \{U_{[a]}\}$ on X . A zero-gerbe corresponds to a principal $\text{U}(1)$ -bundle Γ . Topologically, this bundle is characterised by an element of $H^2(X, \mathbb{Z})$, which, due to the isomorphy between Čech and de Rham cohomology, corresponds to an element F of $H_{\text{dR}}^2(X, \mathbb{Z})$. This element is a curvature two-form taking values in $\mathfrak{u}(1)$ and represents the first Chern class $c_1(\Gamma)$. Application of the Poincaré lemma yields an $\mathfrak{u}(1)$ -valued one-form potential $A_{[a]}$ with $dA_{[a]} = F$ on each patch $U_{[a]}$ and a Čech one-cocycle $f_{[ab]}$ with $df_{[ab]} = A_{[a]} - A_{[b]}$ on intersection of patches $U_{[ab]}$. The exponentials $t_{[ab]} := \exp(f_{[ab]})$ are the transition functions of Γ . Inversely, we can recover the representative F of the first Chern class using a (smooth) partition of unity $\theta = \{\theta_{[a]}\}$ subordinate to the cover \mathfrak{U} : $A_{[a]} = \sum_b d(f_{[ab]}\theta_{[b]})$ and $F = \sum_a d(A_{[a]}\theta_{[a]})$.

³⁷One can equivalently consider the associated line bundle.

³⁸This isomorphism can be derived from the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{E}_X \xrightarrow{\exp} \mathcal{E}_X^* \rightarrow 1$ together with the fact that \mathcal{E}_X is a fine sheaf, which implies $H^n(X, \mathcal{E}_X) = 0$.

In the case of a one-gerbe Γ , we start from a curvature three-form $H \in H_{\text{dR}}^3(X, \mathbb{Z})$ representing the *Dixmier–Douady class* of Γ , the Poincaré lemma leads to a two-form potential $B \in H^1(X, \Omega_X^2)$, a one-form potential $A \in H^2(X, \Omega_X^1)$ and a Čech two-cocycle $f_{[abc]} \in H^3(X, \mathcal{E}^*)$, which satisfy

$$\begin{aligned} H &= dB_{[a]} \text{ on } U_{[a]}, \quad B_{[a]} - B_{[b]} = dA_{[ab]} \text{ on } U_{[ab]}, \\ A_{[ab]} + A_{[bc]} + A_{[ca]} &= df_{[abc]} \text{ on } U_{[abc]}. \end{aligned} \quad (\text{B.2})$$

The set (A, B, H) is also known as the connective structure of the gerbe Γ . The inverse construction of H from f is again performed as above using the partition of unity θ .

The highest gerbe we shall need is a two-gerbe Γ . Here, we start from a curvature four-form³⁹ G representing the characteristic class of Γ . From G , we derive successively three-, two-, and one-form potentials $C \in H^1(X, \Omega_X^3)$, $B \in H^2(X, \Omega_X^2)$, $A \in H^3(X, \Omega_X^1)$. Eventually, we arrive at the Čech three-cocycle $f \in H^3(X, \mathcal{E}_X^*)$.

Note that the notions of pull-back and restriction of an n -gerbe trivially generalise from those of functions and $\text{U}(1)$ -bundles. An n -gerbe is topologically trivial, if the class of its defining Čech cocycle f in $H^{n+1}(X, \mathcal{E}_X^*)$ vanishes. That is, the Čech $(n+1)$ -cocycle is the coboundary of a Čech n -cochain. A holomorphic n -gerbe is a gerbe whose defining $(n+1)$ -cocycle f is an element of $H^{n+1}(X, \mathcal{O}_X^*) \subset H^{n+1}(X, \mathcal{E}_X^*)$. That is, $\bar{\partial}f_{[a_1 \dots a_{n+1}]} = 0$ on each $U_{[a_1 \dots a_{n+1}]}$ where $\bar{\partial}$ is the Dolbeault operator on X . An n -gerbe is holomorphically trivial, if the class of the defining Čech $(n+1)$ -cocycle f is trivial and therefore is the boundary of a holomorphic Čech n -cochain.

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³⁹While the alphabetic order in the common nomenclature for potentials A, B, C matches the order of their degrees, this, unfortunately, does not hold for the corresponding field strengths F, H, G .

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